The spin of an electron can be described with two similar notions. On the one hand with a two-dimensional state vector (complex valued). On the other hand, we identify the spin in space with a three-dimensional spatial vector. We will look at this combination.

Hope I can help you with quantum mechanics.

Dieter Kriesell

The state of a spin in z-direction can be described by two kets, spin up and spin down:

$$|+\rangle \; and \; |-\rangle$$

Note: $|+\rangle \; and \; |-\rangle$ are established notations, omitting the "z".

$|+\rangle \; and \; |-\rangle$ corresponds to an electron with an intrinsic angular momentum up and down:

$$S_z = +\frac{\hbar}{2} \; and \; S_z = -\frac{\hbar}{2}$$

In quantum mechanics any observable corresponds to a Hermitian operator (matrix).

The spin operator $\hat{S}_z$ applied to the state $|+\rangle$ or $|-\rangle$ will reproduce the state, giving back the value of the observable.

Mathematically these are eigenvector and eigenvalue:

$$\hat{S}_z|+\rangle = +\frac{\hbar}{2}|+\rangle, \qquad \hat{S}_z|-\rangle = -\frac{\hbar}{2}|-\rangle$$

The space of states of the electron spin is a two-dimensional complex valued vector space.

A two-dimensional space has a basis of two vectors, ideally an orthonormal basis. We chose:

$$|+\rangle = \begin{pmatrix}1\\0\end{pmatrix}, \qquad |-\rangle = \begin{pmatrix}0\\1\end{pmatrix}$$

Obviously, both build an orthonormal basis of a two-dimensional space:

$$\||+\rangle| = \left|\begin{pmatrix}1\\0\end{pmatrix}\right| = 1 \qquad \||-\rangle| = \left|\begin{pmatrix}0\\1\end{pmatrix}\right| = 1$$

We check the inner product:

$$\langle -|+\rangle = (0,1) \cdot \begin{pmatrix}1\\0\end{pmatrix} = 0$$

Note: in complex valued vector spaces the change from column vector to row vector needs complex conjugation. In the case of matrices this includes transposition and complex conjugation of every matrix. Here the vector has only real components, so complex conjugation is invisible.

In our space of states any state vector $|\psi\rangle$ can be represented by the two basis vectors:

$$|\psi\rangle = c_1|+\rangle + c_2|-\rangle$$

Note: $c_1, c_2 \in \mathbb{C}$

Using our basis, we can write:

$$|\psi\rangle = c_1\begin{pmatrix}1\\0\end{pmatrix} + c_2\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}c_1\\c_2\end{pmatrix}$$

Note: The values of the constants c_1 and c_2 change if the basis changes.

An operator in this two-dimensional, complex valued space of states is represented by a 2×2 Hermitian matrix.

A matrix is called Hermitian if it is identical with its transposed and complex conjugated version.

In its most general form, we write this as:

$$\begin{pmatrix} a+d & b-ic \\ b+ic & a-d \end{pmatrix}$$

Note: $a, b, c, d \in \mathbb{R}$

Example: $a = 1, b = 2, c = 3, d = 4$:

$$\begin{pmatrix} a+d & b-ic \\ b+ic & a-d \end{pmatrix} \to \begin{pmatrix} 5 & 2-i3 \\ 2+i3 & -3 \end{pmatrix}$$

We transpose $\begin{pmatrix} 5 & 2+i3 \\ 2-i3 & -3 \end{pmatrix}$ and take the complex conjugate $\begin{pmatrix} 5 & 2-i3 \\ 2+i3 & -3 \end{pmatrix}$ and get back the original matrix.

Similar to a vector space we find a basis set of matrices any Hermitian 2×2 matrix can be built of:

$$\begin{pmatrix} a+d & b-ic \\ b+ic & a-d \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + d\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

These are the identity matrix $\hat{I}$ and the three Pauli matrices:

$$a \cdot \hat{I} + b \cdot \hat{\sigma}_x + c \cdot \hat{\sigma}_y + d \cdot \hat{\sigma}_z$$

Let us see how Pauli matrices act on our basis vectors:

The operator $\hat{\sigma}_x$, acting on $\vert+\rangle$ and $\vert-\rangle$:	The operator $\hat{\sigma}_y$, acting on $\vert+\rangle$ and $\vert-\rangle$:	**The operator $\boldsymbol{\hat{\sigma}_z}$, acting on $\vert+\rangle$ $\boldsymbol{and}$ $\vert-\rangle$:**
$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ i \end{pmatrix} = i\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -i \\ 0 \end{pmatrix} = -i\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} = -\begin{pmatrix} 0 \\ 1 \end{pmatrix}$
$\hat{\sigma}_x\vert+\rangle = \vert-\rangle$ $\hat{\sigma}_x\vert-\rangle = \vert+\rangle$	$\hat{\sigma}_y\vert+\rangle = i\vert-\rangle$ $\hat{\sigma}_y\vert-\rangle = -i\vert+\rangle$	$\hat{\sigma}_z\vert+\rangle = \vert+\rangle$ $\hat{\sigma}_z\vert-\rangle = -\vert-\rangle$

Note: In quantum mechanics we measure absolute values of observables only, so the minus sign in front of $|-\rangle$ is irrelevant for the result of a measurement. More precisely spoken the minus sign represents an absolute phase and can be ignored for our purposes.

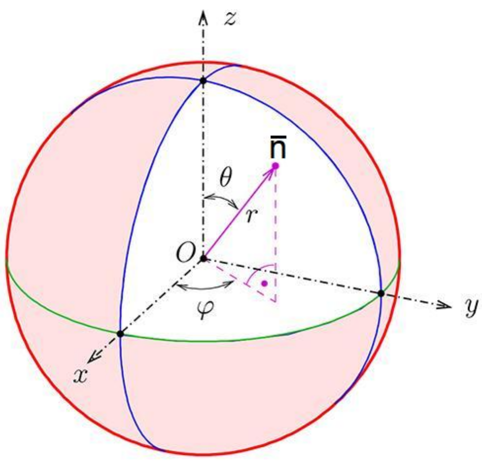
What we need is:

$$\hat{S}_z|+\rangle = \frac{\hbar}{2}|+\rangle, \qquad \hat{S}_z|-\rangle = \frac{\hbar}{2}|-\rangle$$

The appropriate matrix (operator) for this behavior is the operator $\hat{\sigma}_z$:

$$\hat{S}_z = \frac{\hbar}{2}\hat{\sigma}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Note: Mind the absolute phase.



Spin in x-direction

For the z-direction we found the operator and the Pauli matrix. What if the spin is aligned in x- or y-direction?

We will try. We express the x-direction by the $|+\rangle$ and $|-\rangle$ basis.

At this point, we need to carefully distinguish between spatial representation and state space.

We do not want to use another basis in our space of states. We found the Pauli $\hat{\sigma}_z$-matrix fitting for the spin in z-direction, so the Pauli $\hat{\sigma}_x$ should do the job in x-direction:

$$\hat{\sigma}_x \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

We try:

$$\frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\hat{\sigma}_x \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Obviously, the spin in x-direction can be expressed as:

$$\frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The spin in $-x$-direction must be orthogonal. We try the inner product:

$$\frac{1}{\sqrt{2}}(1,1)\cdot\frac{1}{\sqrt{2}}\begin{pmatrix} ? \\ ? \end{pmatrix} = 0 \Longrightarrow \frac{1}{\sqrt{2}}(1,1)\cdot\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0$$

We get the spin $-x$-direction:

$$\frac{1}{\sqrt{2}}|+\rangle - \frac{1}{\sqrt{2}}|-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Note: A minus sign in front of a vector is an absolute phase that can be ignored for our purposes. A minus sing within a vector is a relative phase that matters.

We check:

$$\hat{\sigma}_x \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = -\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

The same works for the spin in y-direction, the Pauli $\hat{\sigma}_y$ should do the job:

$$\hat{\sigma}_y \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

We try:

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{\sqrt{2}} |+\rangle + \frac{i}{\sqrt{2}} |-\rangle$$

We applicate $\hat{\sigma}_y$ onto this vector:

$$\hat{\sigma}_y \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

Obviously, the spin in y-direction can be expressed as:

$$\frac{1}{\sqrt{2}} |+\rangle + \frac{i}{\sqrt{2}} |-\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

The spin in $-y$-direction must be orthogonal. We try the inner product:

$$\frac{1}{\sqrt{2}} (1, -i) \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} ? \\ ? \end{pmatrix} = 0 \Rightarrow \frac{1}{\sqrt{2}} (1, -i) \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} = 0$$

Note: If we transpose a column vector to a row vector we need to complex conjugate it.

We get the spin $-y$-direction:

$$\frac{1}{\sqrt{2}} |+\rangle - \frac{i}{\sqrt{2}} |-\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

We can combine the operators $\hat{S}_x, \hat{S}_y, \hat{S}_z$ to a single compound operator $\hat{S}$:

$$\hat{S} = \frac{\hbar}{2} \left(\hat{\sigma}_x + \hat{\sigma}_y + \hat{\sigma}_z \right) = \frac{\hbar}{2} \begin{pmatrix} 1 & 1-i \\ 1+i & -1 \end{pmatrix}$$

The operator $\hat{S}$ contains a single compound Pauli matrix $\vec{\sigma}$:

$$\vec{\sigma} = \begin{pmatrix} 1 & 1-i \\ 1+i & -1 \end{pmatrix}$$

Result

Our space of states $|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is sufficient to describe the spin in each of the spatial directions x, y and z:

The spin in spatial direction $\pm x$ relates to the state vectors $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

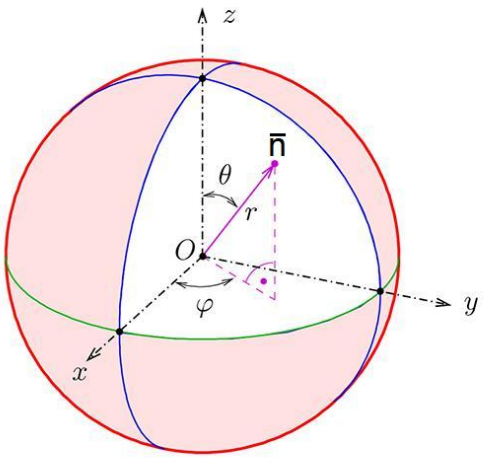
The spin in spatial direction $\pm y$ relates to the state vectors $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$

The spin in spatial direction $\pm z$ relates to the state vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Each of these pairs builds an orthonormal basis of the space of state.

Note: the space of state is a two-dimensional complex vector space. A two-dimensional vector space corresponds to a four-dimensional real vector space and is enough to describe a three-dimensional spatial space.

Spin in any spatial direction



We want to describe the spin in any spatial direction $\vec{n}$ by help of the state vectors. In spherical coordinates $\vec{n}$ can be expressed:

$$\vec{n} = (n_x, n_y, n_z) = (sin(\theta)\cos(\varphi), sin(\theta)sin(\varphi), cos(\theta))$$

We have a spin with three components:

$$\hat{S} = (\hat{S}_x, \hat{S}_y, \hat{S}_z)$$

To get the spin operator $\hat{S}_n$ that points in the spatial direction $\vec{n}$ we build the dot product:

$$\hat{S}_n \coloneqq \hat{S} \cdot \vec{n}$$

More detailed:

$$\hat{S}_n \coloneqq \hat{S} \cdot \vec{n} = n_x\hat{S}_x + n_y\hat{S}_y + n_z\hat{S}_z = \frac{\hbar}{2}\vec{n} \cdot \vec{\sigma}$$

We do this explicitly, using:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\hat{S}_n = \frac{\hbar}{2}\vec{n} \cdot \vec{\sigma} = \frac{\hbar}{2}\left(n_x\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + n_y\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + n_z\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) =$$

$$\frac{\hbar}{2}\left(\begin{pmatrix} 0 & sin(\theta)\cos(\varphi) \\ sin(\theta)\cos(\varphi) & 0 \end{pmatrix} + \begin{pmatrix} 0 & -isin(\theta)sin(\varphi) \\ isin(\theta)sin(\varphi) & 0 \end{pmatrix} + \begin{pmatrix} cos(\theta) & 0 \\ 0 & -cos(\theta) \end{pmatrix}\right) =$$

$$\frac{\hbar}{2}\begin{pmatrix} cos(\theta) & sin(\theta)\cos(\varphi) - isin(\theta)sin(\varphi) \\ sin(\theta)\cos(\varphi) + isin(\theta)sin(\varphi) & -cos(\theta) \end{pmatrix} =$$

$$\frac{\hbar}{2}\begin{pmatrix} cos(\theta) & sin(\theta)(\cos(\varphi) - isin(\varphi)) \\ sin(\theta)(\cos(\varphi) + isin(\varphi)) & -cos(\theta) \end{pmatrix} =$$

$$\frac{\hbar}{2}\begin{pmatrix} cos(\theta) & sin(\theta)e^{-i\varphi} \\ sin(\theta)e^{i\varphi} & -cos(\theta) \end{pmatrix}$$

We get:

$$\hat{S}_n = \frac{\hbar}{2}\begin{pmatrix} cos(\theta) & sin(\theta)e^{-i\varphi} \\ sin(\theta)e^{i\varphi} & -cos(\theta) \end{pmatrix}$$

A quick check shows that this is compatible with our Pauli matrices.

The x-axis we get by the angles $\theta = \frac{\pi}{2}$ and $\varphi = 0$. With this we have:	$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
The y-axis we get by the angles $\theta = \frac{\pi}{2}$ and $\varphi = \frac{\pi}{2}$. With this we have:	$\hat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$
The z-axis we get by the angles $\theta = 0$ and $\varphi = "any\ value"$. With this we have:	$\hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

We need the eigenvectors and eigenvalues of $\hat{S}_n$.

The eigenvalue equation of $\hat{S}_n$:

$$det\left(\frac{\hbar}{2}\begin{pmatrix} cos(\theta)-\lambda & sin(\theta)e^{-i\varphi} \\ sin(\theta)e^{i\varphi} & -cos(\theta)-\lambda \end{pmatrix}\right)=0$$

$$\frac{\hbar}{2}\Big((cos(\theta)-\lambda)(-cos(\theta)-\lambda)-\big(sin(\theta)e^{-i\varphi}\big)\big(sin(\theta)e^{i\varphi}\big)\Big)=0$$

$$\frac{\hbar}{2}\big((-\cos(\theta)^2-\lambda\cdot cos(\theta)+\lambda\cdot cos(\theta)+\lambda^2)-sin(\theta)^2\big)=0$$

$$\frac{\hbar}{2}\big(\lambda^2-(\cos(\theta)^2+sin(\theta)^2)\big)=0$$

$$\lambda=\pm 1 \rightarrow eigenvalues\ \pm\frac{\hbar}{2}$$

The corresponding eigenvectors of $\hat{S}_n$ are:

$$|+n\rangle=\begin{pmatrix} cos\left(\frac{\theta}{2}\right) \\ sin\left(\frac{\theta}{2}\right)e^{i\varphi} \end{pmatrix}=cos\left(\frac{\theta}{2}\right)|+\rangle+sin\left(\frac{\theta}{2}\right)e^{i\varphi}|-\rangle$$

$$|-n\rangle=\begin{pmatrix} sin\left(\frac{\theta}{2}\right) \\ -cos\left(\frac{\theta}{2}\right)e^{i\varphi} \end{pmatrix}=sin\left(\frac{\theta}{2}\right)|+\rangle-cos\left(\frac{\theta}{2}\right)e^{i\varphi}|-\rangle$$

$|+n\rangle$ and $|-n\rangle$ build an orthonormal basis of the space of state, expressing the Spin in direction $\vec{n}$.

We check $\hat{S}_n\|+n\rangle=\frac{\hbar}{2}\|+n\rangle$:	We check $\hat{S}_n\|-n\rangle=-\frac{\hbar}{2}\|-n\rangle$:
$\frac{\hbar}{2}\begin{pmatrix} cos(\theta) & sin(\theta)e^{-i\varphi} \\ sin(\theta)e^{i\varphi} & -cos(\theta) \end{pmatrix}\begin{pmatrix} cos\left(\frac{\theta}{2}\right) \\ sin\left(\frac{\theta}{2}\right)e^{i\varphi} \end{pmatrix}=$ $\frac{\hbar}{2}\begin{pmatrix} cos(\theta)cos\left(\frac{\theta}{2}\right)+sin(\theta)e^{-i\varphi}sin\left(\frac{\theta}{2}\right)e^{i\varphi} \\ sin(\theta)e^{i\varphi}cos\left(\frac{\theta}{2}\right)-cos(\theta)sin\left(\frac{\theta}{2}\right)e^{i\varphi} \end{pmatrix}$	$\frac{\hbar}{2}\begin{pmatrix} cos(\theta) & sin(\theta)e^{-i\varphi} \\ sin(\theta)e^{i\varphi} & -cos(\theta) \end{pmatrix}\begin{pmatrix} sin\left(\frac{\theta}{2}\right) \\ -cos\left(\frac{\theta}{2}\right)e^{i\varphi} \end{pmatrix}=$ $\frac{\hbar}{2}\begin{pmatrix} cos(\theta)sin\left(\frac{\theta}{2}\right)-sin(\theta)e^{-i\varphi}cos\left(\frac{\theta}{2}\right)e^{i\varphi} \\ sin(\theta)e^{i\varphi}sin\left(\frac{\theta}{2}\right)+cos(\theta)cos\left(\frac{\theta}{2}\right)e^{i\varphi} \end{pmatrix}$
We use trigonometric identity:	
$cos(\theta)=cos^2\left(\frac{\theta}{2}\right)-sin^2\left(\frac{\theta}{2}\right),\qquad sin(\theta)=2sin\left(\frac{\theta}{2}\right)cos\left(\frac{\theta}{2}\right)$	
We get for the first line:	
$cos(\theta)cos\left(\frac{\theta}{2}\right)+sin(\theta)sin\left(\frac{\theta}{2}\right)=$ $\left(cos^2\left(\frac{\theta}{2}\right)-sin^2\left(\frac{\theta}{2}\right)\right)cos\left(\frac{\theta}{2}\right)$ $+2sin\left(\frac{\theta}{2}\right)cos\left(\frac{\theta}{2}\right)sin\left(\frac{\theta}{2}\right)=$ $cos^3\left(\frac{\theta}{2}\right)-sin^2\left(\frac{\theta}{2}\right)cos\left(\frac{\theta}{2}\right)+2sin^2\left(\frac{\theta}{2}\right)cos\left(\frac{\theta}{2}\right)=$ $cos^3\left(\frac{\theta}{2}\right)+sin^2\left(\frac{\theta}{2}\right)cos\left(\frac{\theta}{2}\right)=$ $\left(cos^2\left(\frac{\theta}{2}\right)+sin^2\left(\frac{\theta}{2}\right)\right)cos\left(\frac{\theta}{2}\right)=$	$cos(\theta)sin\left(\frac{\theta}{2}\right)-sin(\theta)e^{-i\varphi}cos\left(\frac{\theta}{2}\right)e^{i\varphi}=$ $\left(cos^2\left(\frac{\theta}{2}\right)-sin^2\left(\frac{\theta}{2}\right)\right)sin\left(\frac{\theta}{2}\right)$ $-2sin\left(\frac{\theta}{2}\right)cos\left(\frac{\theta}{2}\right)cos\left(\frac{\theta}{2}\right)=$ $cos^2\left(\frac{\theta}{2}\right)sin\left(\frac{\theta}{2}\right)-sin^3\left(\frac{\theta}{2}\right)-2sin\left(\frac{\theta}{2}\right)cos^2\left(\frac{\theta}{2}\right)=$ $\left(cos^2\left(\frac{\theta}{2}\right)-sin^2\left(\frac{\theta}{2}\right)-2cos^2\left(\frac{\theta}{2}\right)\right)sin\left(\frac{\theta}{2}\right)=$ $\left(-cos^2\left(\frac{\theta}{2}\right)-sin^2\left(\frac{\theta}{2}\right)\right)sin\left(\frac{\theta}{2}\right)=$

<table>
<tr><td>$cos\left(\frac{\theta}{2}\right)$</td><td>$-sin\left(\frac{\theta}{2}\right)$</td></tr>
<tr><td colspan="2">We get for the second line:</td></tr>
<tr><td>$$sin(\theta)e^{i\varphi}cos\left(\frac{\theta}{2}\right) - cos(\theta)sin\left(\frac{\theta}{2}\right)e^{i\varphi} =$$ $$2sin\left(\frac{\theta}{2}\right)cos\left(\frac{\theta}{2}\right)e^{i\varphi}cos\left(\frac{\theta}{2}\right) - \left(cos^2\left(\frac{\theta}{2}\right) - sin^2\left(\frac{\theta}{2}\right)\right)sin\left(\frac{\theta}{2}\right)e^{i\varphi} =$$ $$2sin\left(\frac{\theta}{2}\right)cos^2\left(\frac{\theta}{2}\right)e^{i\varphi} - \left(cos^2\left(\frac{\theta}{2}\right)sin\left(\frac{\theta}{2}\right) - sin^3\left(\frac{\theta}{2}\right)\right)e^{i\varphi} =$$ $$\left(2sin\left(\frac{\theta}{2}\right)cos^2\left(\frac{\theta}{2}\right) - cos^2\left(\frac{\theta}{2}\right)sin\left(\frac{\theta}{2}\right) + sin^3\left(\frac{\theta}{2}\right)\right)e^{i\varphi} =$$ $$\left(cos^2\left(\frac{\theta}{2}\right)sin\left(\frac{\theta}{2}\right) + sin^3\left(\frac{\theta}{2}\right)\right)e^{i\varphi} =$$ $$sin\left(\frac{\theta}{2}\right)\left(cos^2\left(\frac{\theta}{2}\right) + sin^2\left(\frac{\theta}{2}\right)\right)e^{i\varphi} =$$ $$sin\left(\frac{\theta}{2}\right)e^{i\varphi}$$</td><td>$$sin(\theta)e^{i\varphi}sin\left(\frac{\theta}{2}\right) + cos(\theta)cos\left(\frac{\theta}{2}\right)e^{i\varphi}$$ $$2sin\left(\frac{\theta}{2}\right)cos\left(\frac{\theta}{2}\right)e^{i\varphi}sin\left(\frac{\theta}{2}\right) + \left(cos^2\left(\frac{\theta}{2}\right) - sin^2\left(\frac{\theta}{2}\right)\right)cos\left(\frac{\theta}{2}\right)e^{i\varphi} =$$ $$2sin^2\left(\frac{\theta}{2}\right)cos\left(\frac{\theta}{2}\right)e^{i\varphi} + \left(cos^3\left(\frac{\theta}{2}\right) - sin^2\left(\frac{\theta}{2}\right)cos\left(\frac{\theta}{2}\right)\right)e^{i\varphi} =$$ $$\left(2sin^2\left(\frac{\theta}{2}\right)cos\left(\frac{\theta}{2}\right) + cos^3\left(\frac{\theta}{2}\right) - sin^2\left(\frac{\theta}{2}\right)cos\left(\frac{\theta}{2}\right)\right)e^{i\varphi} =$$ $$\left(2sin^2\left(\frac{\theta}{2}\right) + cos^2\left(\frac{\theta}{2}\right) - sin^2\left(\frac{\theta}{2}\right)\right)cos\left(\frac{\theta}{2}\right)e^{i\varphi} =$$ $$\left(sin^2\left(\frac{\theta}{2}\right) + cos^2\left(\frac{\theta}{2}\right)\right)cos\left(\frac{\theta}{2}\right)e^{i\varphi} =$$ $$cos\left(\frac{\theta}{2}\right)e^{i\varphi}$$</td></tr>
<tr><td colspan="2">In summa:</td></tr>
<tr><td>$$\frac{\hbar}{2}\begin{pmatrix} cos(\theta)cos\left(\frac{\theta}{2}\right) + sin(\theta)e^{-i\varphi}sin\left(\frac{\theta}{2}\right)e^{i\varphi} \\ sin(\theta)e^{i\varphi}cos\left(\frac{\theta}{2}\right) - cos(\theta)sin\left(\frac{\theta}{2}\right)e^{i\varphi} \end{pmatrix} =$$ $$\frac{\hbar}{2}\begin{pmatrix} cos\left(\frac{\theta}{2}\right) \\ sin\left(\frac{\theta}{2}\right)e^{i\varphi} \end{pmatrix} = \frac{\hbar}{2}\lvert +n\rangle$$</td><td>$$\frac{\hbar}{2}\begin{pmatrix} cos(\theta)cos\left(\frac{\theta}{2}\right) + sin(\theta)e^{-i\varphi}sin\left(\frac{\theta}{2}\right)e^{i\varphi} \\ sin(\theta)e^{i\varphi}cos\left(\frac{\theta}{2}\right) - cos(\theta)sin\left(\frac{\theta}{2}\right)e^{i\varphi} \end{pmatrix}$$ $$= \frac{\hbar}{2}\begin{pmatrix} -sin\left(\frac{\theta}{2}\right) \\ cos\left(\frac{\theta}{2}\right)e^{i\varphi} \end{pmatrix} = -\frac{\hbar}{2}\lvert -n\rangle$$</td></tr>
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We check the inner product:

$$\langle +n \vert -n\rangle =$$

$$\left(cos\left(\frac{\theta}{2}\right), sin\left(\frac{\theta}{2}\right)e^{-i\varphi}\right) \cdot \begin{pmatrix} sin\left(\frac{\theta}{2}\right) \\ -cos\left(\frac{\theta}{2}\right)e^{i\varphi} \end{pmatrix} =$$

$$cos\left(\frac{\theta}{2}\right)sin\left(\frac{\theta}{2}\right) - sin\left(\frac{\theta}{2}\right)e^{-i\varphi}cos\left(\frac{\theta}{2}\right)e^{i\varphi} =$$

$$cos\left(\frac{\theta}{2}\right)sin\left(\frac{\theta}{2}\right) - sin\left(\frac{\theta}{2}\right)cos\left(\frac{\theta}{2}\right) = 0$$

Note: To transpose a complex vector, we need to complex conjugate it.

We check $\langle +n | +n \rangle = 1$:

$$\left(cos\left(\frac{\theta}{2}\right), sin\left(\frac{\theta}{2}\right)e^{-i\varphi}\right) \cdot \begin{pmatrix} cos\left(\frac{\theta}{2}\right) \\ sin\left(\frac{\theta}{2}\right)e^{i\varphi} \end{pmatrix} = cos^2\left(\frac{\theta}{2}\right) + sin^2\left(\frac{\theta}{2}\right) = 1$$

We check $\langle -n | -n \rangle = 1$:

$$\left(sin\left(\frac{\theta}{2}\right), -cos\left(\frac{\theta}{2}\right)e^{-i\varphi}\right) \cdot \begin{pmatrix} sin\left(\frac{\theta}{2}\right) \\ -cos\left(\frac{\theta}{2}\right)e^{i\varphi} \end{pmatrix} = sin^2\left(\frac{\theta}{2}\right) + cos^2\left(\frac{\theta}{2}\right) = 1$$

This seems to be a veritable orthonormal basis.

Note: $| +n \rangle$ and $| -n \rangle$ in literature often are written as $| + \rangle_n$ and $| - \rangle_n$.