Alice has a qubit $|\psi\rangle$ she wants to transmit to Bob. Alice don't know the state of this qubit.

Alice is unable to physically send her qubit $|\psi\rangle$to Bob (say, by Fed-Ex …).

Alice is able to send classical information.

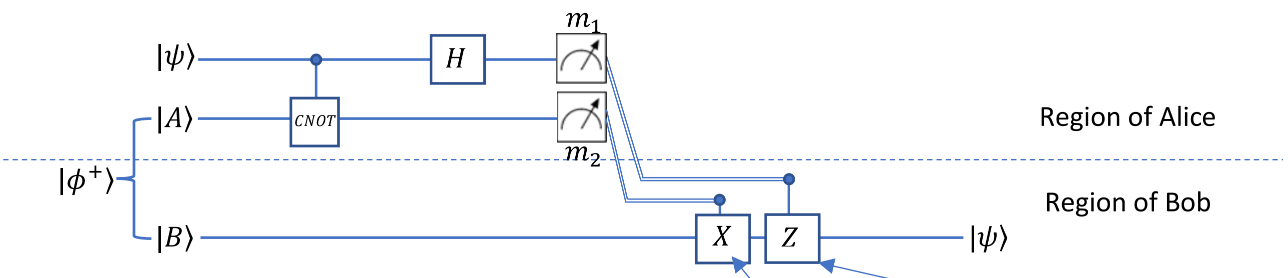
Alice and Bob share an entangled qubit, $|\phi^+\rangle$, one of the Bell states:

Note: This is an experimental text that may contain errors.

$$|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

Alice possesses the second qubit of $|\phi^+\rangle$, Bob the first one.

Now the following procedure takes place:




Note:

Single lines denote qubits

Double lines denote classical bits

$H$ Denote the Hadamard gate

$X$ Denote a bit flip (the Pauli x-gate)

$Z$ Denote a phase flip (the Pauli z-gate)

$CNOT$ Denote a controlled Not gate

Note: The choice of the gates on the line $|B\rangle$ will become clear at the end of the process.

The procedure goes as follows.

First Alice:

- Alice performs a $CNOT$ from $|\psi\rangle$ to $|A\rangle$
- Alice performs a Hadamard on $|\psi\rangle$
- Alice measures $|A\rangle$ and $|\psi\rangle$ obtaining the classical bits $m_1$ and $m_2$
- Alice sends classical bits $m_1$ and $m_2$ to Bob

Then Bob:

- Bob first performs a $CNOT$ controlled by $m_2$ on $|B\rangle$
- Bob second performs a phase flip controlled by $m_1$ on $|B\rangle$

The operation of Bob:

$Id$ $\quad if\ ab = 00$
$Z$ $\quad if\ ab = 01$
$X$ $\quad if\ ab = 10$
$ZX$ $\quad if\ ab = 11$

The result is the teleportation:

$$|B\rangle\ becomes\ |\psi\rangle$$

Let us take a closer look at the scenario. The initial state of the three qubits $|\psi\rangle$ and $|\phi^+\rangle$ we name $|\pi_0\rangle$.

In detail:

$|\pi_0\rangle = |B\rangle|A\rangle|\psi\rangle$ or $|BA\psi\rangle$

The qubit $|\psi\rangle$ of Alice is in the state $\alpha|0\rangle + \beta|1\rangle$:

We get:

$$|\pi_0\rangle = |\phi^+\rangle(\alpha|0\rangle + \beta|1\rangle)$$

We expand $|\phi^+\rangle$:

$$|\pi_0\rangle = \left(\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)\right)(\alpha|0\rangle + \beta|1\rangle) =$$

$$\frac{\alpha|000\rangle + \alpha|110\rangle + \beta|001\rangle + \beta|111\rangle}{\sqrt{2}}$$

We apply the $CNOT$ on $|\pi_0\rangle$ and get $|\pi_1\rangle$.

Note that the $CNOT$ is controlled by the last qubit and acts on the second qubit.

$$|\pi_1\rangle = \frac{\alpha|000\rangle + \alpha|110\rangle + \beta|011\rangle + \beta|101\rangle}{\sqrt{2}}$$

We rewrite:

$$|\pi_1\rangle = \frac{\alpha|000\rangle + \alpha|110\rangle + \beta|011\rangle + \beta|101\rangle}{\sqrt{2}} = \frac{\alpha|00\rangle|0\rangle + \alpha|11\rangle|0\rangle + \beta|01\rangle|1\rangle + \beta|10\rangle|1\rangle}{\sqrt{2}} =$$

$$\frac{(\alpha|00\rangle + \alpha|11\rangle)|0\rangle + (\beta|01\rangle + \beta|10\rangle)|1\rangle}{\sqrt{2}}$$

We apply the Hadamard on $|\psi\rangle$:

$$H|\pi_1\rangle = \frac{(\alpha|00\rangle + \alpha|11\rangle)\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) + (\beta|01\rangle + \beta|10\rangle)\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)}{\sqrt{2}} =$$

$$\frac{(\alpha|00\rangle + \alpha|11\rangle)(|0\rangle + |1\rangle) + (\beta|01\rangle + \beta|10\rangle)(|0\rangle - |1\rangle)}{2} =$$

$$\frac{\alpha|000\rangle + \alpha|001\rangle + \alpha|110\rangle + \alpha|111\rangle + \beta|010\rangle - \beta|011\rangle + \beta|100\rangle - \beta|101\rangle}{2} = \pi_2$$

This is the state after Alice applied the $CNOT$ and the Hadamard onto her two qubits, $|\psi\rangle$ and her half of $|\phi^+\rangle$.

We remember that constants can float freely through tensor products and regroup $\pi_2$:

$$(\alpha|000\rangle + \alpha|001\rangle + \alpha|110\rangle + \alpha|111\rangle + \beta|010\rangle - \beta|011\rangle + \beta|100\rangle - \beta|101\rangle) \rightarrow$$

$$\frac{1}{2}(\alpha|000\rangle + \beta|100\rangle + \alpha|001\rangle - \beta|101\rangle + \alpha|110\rangle + \beta|010\rangle + \alpha|111\rangle - \beta|011\rangle) =$$

$$\frac{1}{2}\big((\alpha|0\rangle + \beta|1\rangle)|00\rangle + (\alpha|0\rangle - \beta|1\rangle)|01\rangle + (\alpha|1\rangle + \beta|0\rangle)|10\rangle + (\alpha|1\rangle - \beta|0\rangle)|11\rangle\big)$$

We remember the ordering: $|\pi_2\rangle = |B\rangle|A\rangle|\psi\rangle$

The effect of the reordering: It seems as if $|B\rangle$ has changed and depends upon $\alpha$ and $\beta$.

Now Alice measures. We take a look at the possible results of $m_1$ and $m_2$:

$$prob(m_1m_2 = 00) = \frac{1}{4}\|\alpha|0\rangle + \beta|1\rangle\|^2 = \frac{1}{4}$$
$$prob(m_1m_2 = 01) = \frac{1}{4}\|\alpha|0\rangle - \beta|1\rangle\|^2 = \frac{1}{4}$$
$$prob(m_1m_2 = 10) = \frac{1}{4}\|\alpha|1\rangle + \beta|0\rangle\|^2 = \frac{1}{4}$$
$$prob(m_1m_2 = 11) = \frac{1}{4}\|\alpha|1\rangle + \beta|0\rangle\|^2 = \frac{1}{4}$$

All outcomes on Alice' side has equal probability. We have no dependency on $\alpha$ and $\beta$ meaning that the choice of $\alpha$ and $\beta$ have no impact on the result of the measurement.

Note: After the measurement the qubit of Alice is destroyed so we will get no conflict with the no-clone theorem.

We build a list of all measurement outcomes Alice takes and the conditional state of $|B\rangle$~~$|A\rangle|\psi\rangle$~~:

| $m_1m_2$ | probability | Conditional state $|B\rangle$~~$|A\rangle|\psi\rangle$~~ |
|---|---|---|
| $00$ | $\frac{1}{4}$ | $(\alpha|0\rangle + \beta|1\rangle)$ |
| $01$ | $\frac{1}{4}$ | $(\alpha|0\rangle - \beta|1\rangle)$ |
| $10$ | $\frac{1}{4}$ | $(\alpha|1\rangle + \beta|0\rangle)$ |
| $11$ | $\frac{1}{4}$ | $(\alpha|1\rangle - \beta|0\rangle)$ |

Note: No information about $\alpha$ and $\beta$ could be gained by measurement on Bob's side.

Now Bob performs his operations on qubit $|B\rangle$ according to the measurement results he got from Alice. We add this to our table:

| $m_1m_2$ | probability | Conditional state $|B\rangle$~~$|A\rangle|\psi\rangle$~~ | Operation on $|B\rangle$ | Final state of $|B\rangle$ |
|---|---|---|---|---|
| $00$ | $\frac{1}{4}$ | $(\alpha|0\rangle + \beta|1\rangle)$ | $Id$ | $\alpha|0\rangle + \beta|1\rangle$ |
| $01$ | $\frac{1}{4}$ | $(\alpha|0\rangle - \beta|1\rangle)$ | $Z$ | $\alpha|0\rangle + \beta|1\rangle$ |
| $10$ | $\frac{1}{4}$ | $(\alpha|1\rangle + \beta|0\rangle)$ | $X$ | $\alpha|0\rangle + \beta|1\rangle$ |
| $11$ | $\frac{1}{4}$ | $(\alpha|1\rangle - \beta|0\rangle)$ | $Z$ $X$ | $\alpha|0\rangle + \beta|1\rangle$ |

We see that Bob's qubit $|B\rangle$ now is in the state Alice's qubit $|\psi\rangle$ was. The state of the qubit has been "teleported", $|\psi\rangle$ and the part $|A\rangle$ of $|\phi^+\rangle$ are destroyed.

Note: You may find more information at:

https://learning.quantum.ibm.com/course/basics-of-quantum-information/entanglement-in-action

So far, we have worked on the conceptual level with bras and kets.

How does this look like if we work with state vectors? Let us try this too.

We have in total three qubits so our state vector will have 8 dimensions.

Note: You may refer to "Teleportation_1" if you want to derive the explicit forms of the gates $CNOT$, $X$ and $Z$.

Using the same names as at the conceptual level we have:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \binom{\alpha}{\beta}$$

$$|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$$

Note: $|0\rangle \otimes |0\rangle = \binom{1}{0} \otimes \binom{1}{0} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$, $|1\rangle \otimes |1\rangle = \binom{0}{1} \otimes \binom{0}{1} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$

$$|\pi_0\rangle = |\psi\rangle|\phi^+\rangle = |\psi\rangle \otimes |\phi^+\rangle =$$

$$\frac{1}{\sqrt{2}}\binom{\alpha}{\beta} \otimes \begin{pmatrix}1\\0\\0\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}\alpha\\0\\0\\\alpha\\\beta\\0\\0\\\beta\end{pmatrix}$$

The CNOT from line one to line two:

$$\begin{pmatrix}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&1\\0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\end{pmatrix}$$

We apply the CNOT from line one to line two:

$$\begin{pmatrix}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&1\\0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}\alpha\\0\\0\\\alpha\\\beta\\0\\0\\\beta\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}\alpha\\0\\0\\\alpha\\0\\\beta\\\beta\\0\end{pmatrix}$$

Hadamard on line one:

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{pmatrix}$$

We apply the Hadamard gate onto line one:

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} \alpha \\ 0 \\ 0 \\ \alpha \\ 0 \\ \beta \\ \beta \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \alpha \\ \alpha \\ \beta \\ -\beta \\ \beta \\ -\beta \\ \alpha \\ \alpha \end{pmatrix}$$

We compare this with the conceptual level, there we had:

$$\frac{1}{2}\big((\alpha|0\rangle + \beta|1\rangle)|00\rangle + (\alpha|0\rangle - \beta|1\rangle)|01\rangle + (\alpha|1\rangle + \beta|0\rangle)|10\rangle + (\alpha|1\rangle - \beta|0\rangle)|11\rangle\big)$$

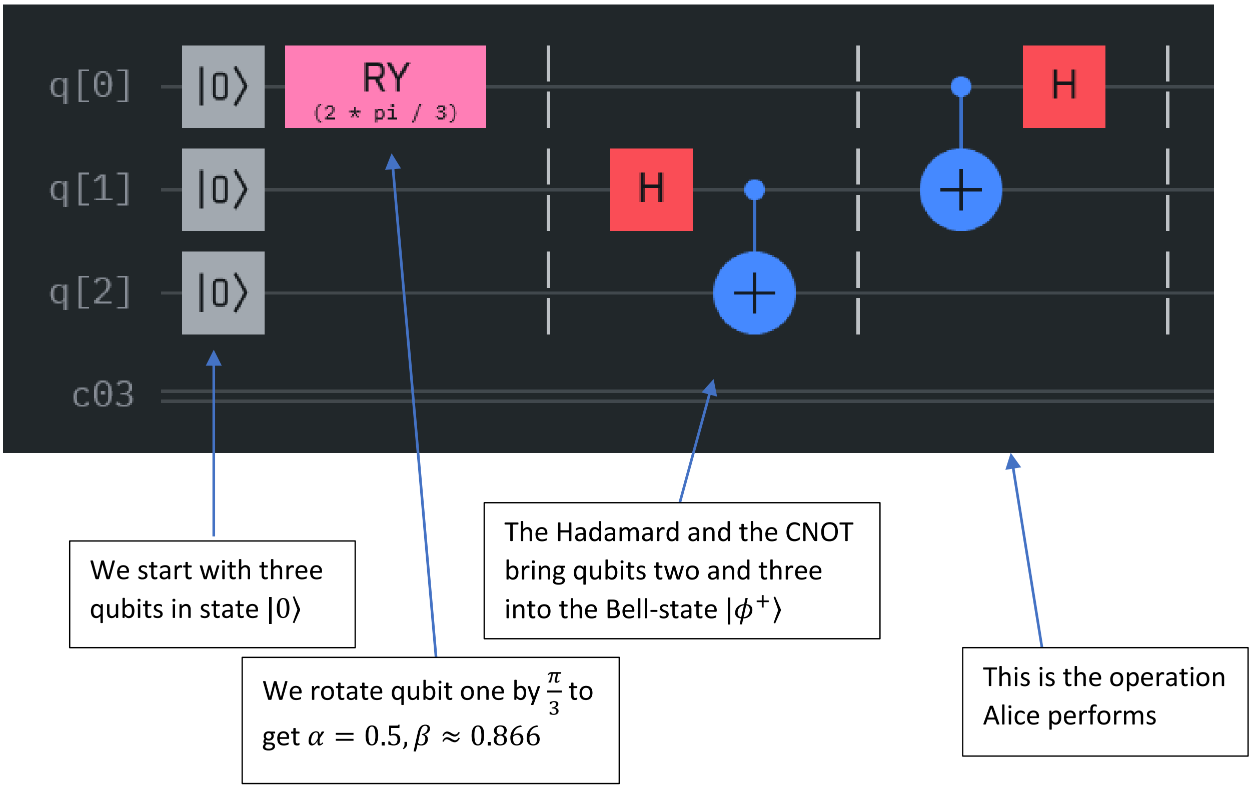
We see the correspondence:

| | | | | | |
|---|---|---|---|---|---|
| $\frac{1}{2}\begin{pmatrix} \alpha \\ \alpha \\ \beta \\ -\beta \\ \beta \\ -\beta \\ \alpha \\ \alpha \end{pmatrix}$ | $\begin{pmatrix} \alpha \\ 0 \\ 0 \\ 0 \end{pmatrix}$ $\begin{pmatrix} \beta \\ 0 \\ 0 \\ 0 \end{pmatrix}$ | collapse to $\|00\rangle$ $\rightarrow \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ remains | $\frac{1}{2}\begin{pmatrix} \alpha \\ \alpha \\ \beta \\ -\beta \\ \beta \\ -\beta \\ \alpha \\ \alpha \end{pmatrix}$ | $\begin{pmatrix} 0 \\ \alpha \\ 0 \\ 0 \end{pmatrix}$ $\begin{pmatrix} 0 \\ -\beta \\ 0 \\ 0 \end{pmatrix}$ | collapse to $\|01\rangle$ $\rightarrow \begin{pmatrix} \alpha \\ -\beta \end{pmatrix}$ remains |
| $\frac{1}{2}\begin{pmatrix} \alpha \\ \alpha \\ \beta \\ -\beta \\ \beta \\ -\beta \\ \alpha \\ \alpha \end{pmatrix}$ | $\begin{pmatrix} 0 \\ 0 \\ \beta \\ 0 \end{pmatrix}$ $\begin{pmatrix} 0 \\ 0 \\ \alpha \\ 0 \end{pmatrix}$ | collapse to $\|10\rangle$ $\rightarrow \begin{pmatrix} \beta \\ \alpha \end{pmatrix}$ remains | $\frac{1}{2}\begin{pmatrix} \alpha \\ \alpha \\ \beta \\ -\beta \\ \beta \\ -\beta \\ \alpha \\ \alpha \end{pmatrix}$ | $\begin{pmatrix} 0 \\ 0 \\ 0 \\ -\beta \end{pmatrix}$ $\begin{pmatrix} 0 \\ 0 \\ 0 \\ \alpha \end{pmatrix}$ | collapse to $\|11\rangle$ $\rightarrow \begin{pmatrix} -\beta \\ \alpha \end{pmatrix}$ remains |

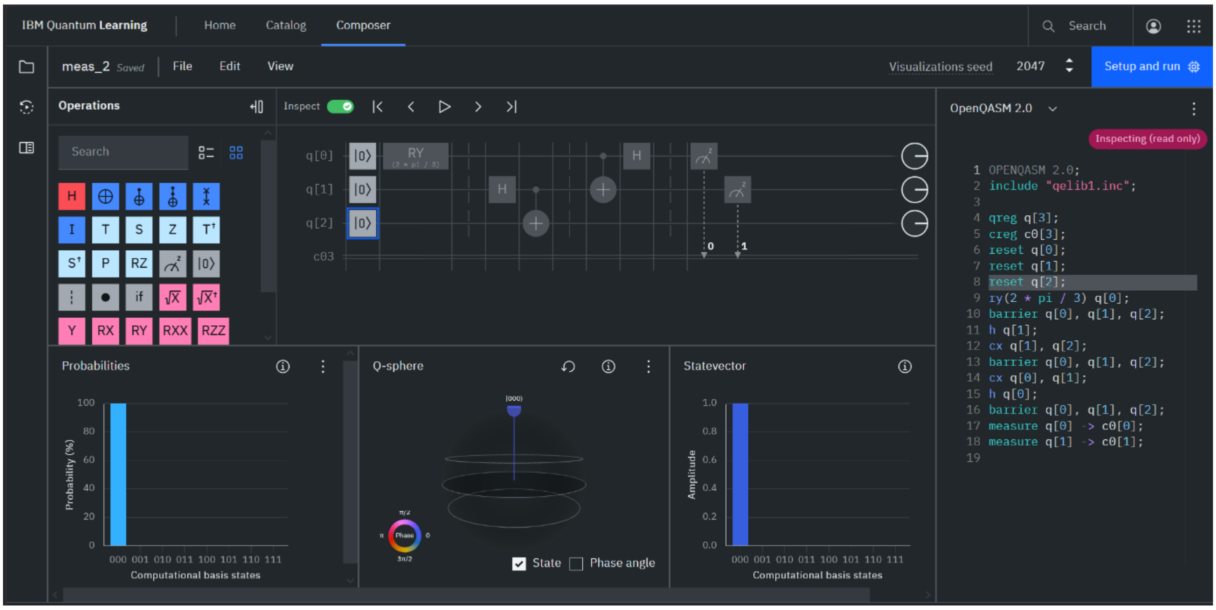
Measuring collapses the state vector leaving different residues. They are determined so Bob can choose the gates he must use to restore the original vector $|\psi\rangle$ of Alice.

Let us see how this looks like in the IBM Composer.

We use the circuit:

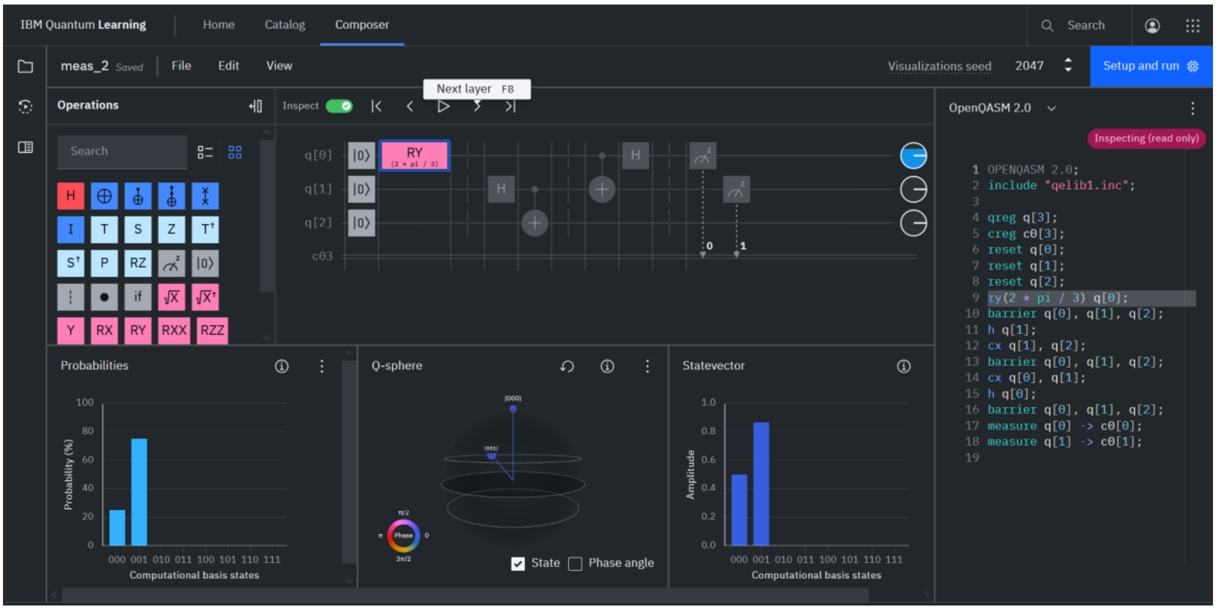



We check the circuit step by step. Note that in the pictures we have IBM-notation, the upper qubit is right.

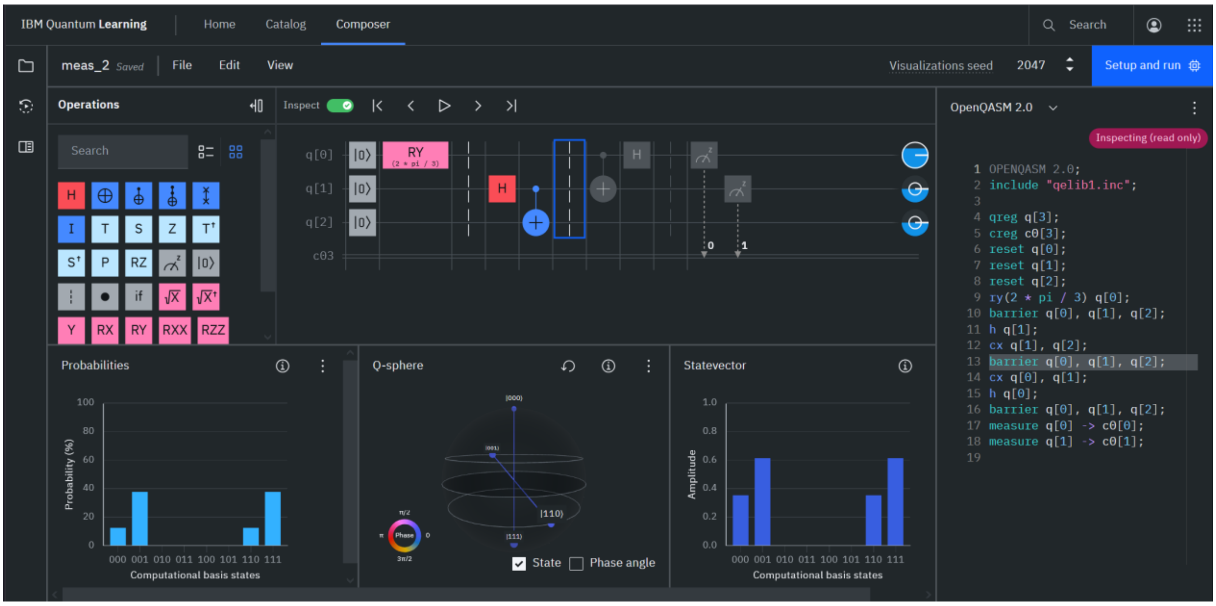



We see the state vector is triple $|000\rangle$.

We rotate the first qubit by 60° resp. $\frac{\pi}{3}$:




We see the first qubit has probability amplitude 0.5 for $\alpha$ and 0.866 for $\beta$. The probabilities then are 25% resp. 75%.

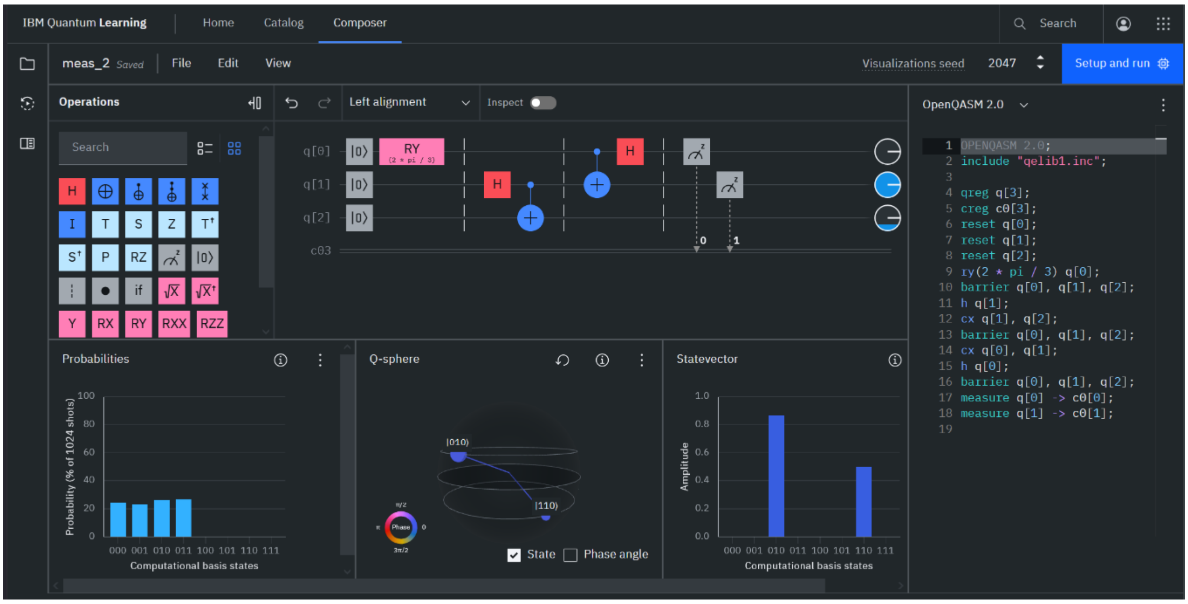



We produced the Bell-state $|\phi^+\rangle$, the lower two qubits are in the state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Note that the probability amplitude is mixed up with the factor belonging to the first qubit.

Now Alice performs her operation, a CNOT and a Hadamard:


We get a superposition of all three qubits.

Now Alice measures the first two qubits:



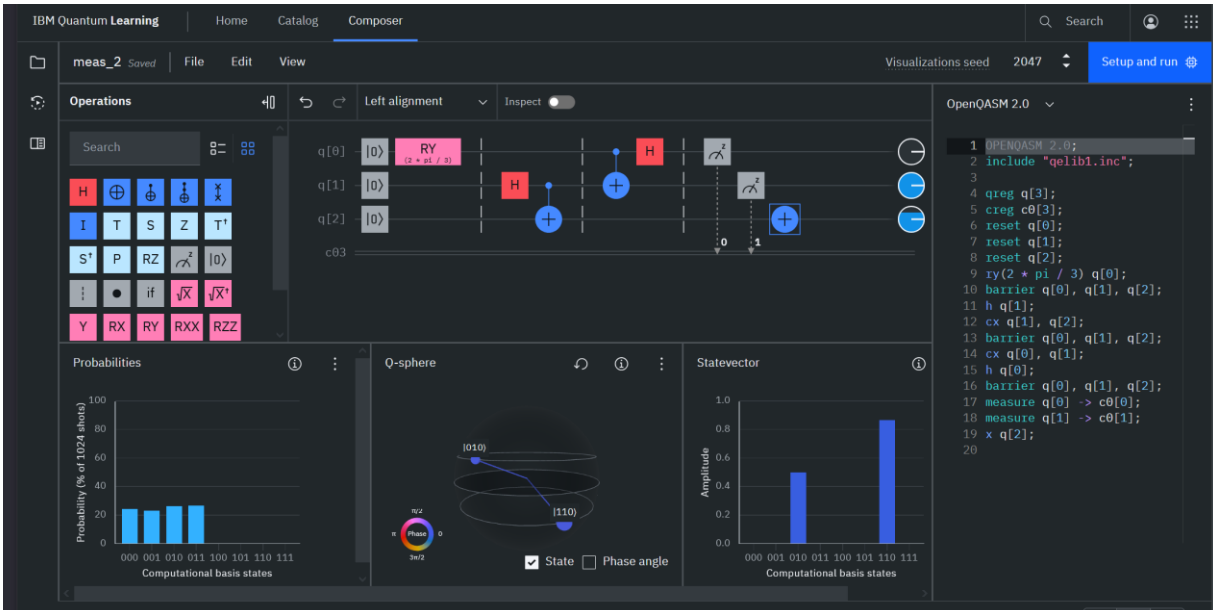

The simulator shows the third case:

... this is my assumption ...

| $m_1 m_2$ | probability | Conditional state $\vert B\rangle$ | Operation on $\vert B\rangle$ | Final state of $\vert B\rangle$ |
|---|---|---|---|---|
| 00 | $\frac{1}{4}$ | $(\alpha\vert 0\rangle + \beta\vert 1\rangle)\vert 00\rangle$ | $Id$ | $\alpha\vert 0\rangle + \beta\vert 1\rangle$ |
| 01 | $\frac{1}{4}$ | $(\alpha\vert 0\rangle - \beta\vert 1\rangle)\vert 01\rangle$ | $Z$ | $\alpha\vert 0\rangle + \beta\vert 1\rangle$ |
| **10** | $\boldsymbol{\frac{1}{4}}$ | $\boldsymbol{(\alpha\vert 1\rangle + \beta\vert 0\rangle)\vert 10\rangle}$ | $X$ | $\boldsymbol{\alpha\vert 0\rangle + \beta\vert 1\rangle}$ |
| 11 | $\frac{1}{4}$ | $(\alpha\vert 1\rangle - \beta\vert 0\rangle)\vert 11\rangle$ | $Z$ $X$ | $\alpha\vert 0\rangle + \beta\vert 1\rangle$ |

In this case Bob needs an additional NOT-gate to get the original qubit $|\psi\rangle$ of Alice.

We insert the NOT-gate:




We see that the state vector has only two components in the first entry, the last qubit, with probability amplitudes 0.5 for qubit $|0\rangle$ and 0.866 for qubit $|1\rangle$. This is the state the first qubit was in at the beginning of the process.

You can use the composer and try different value for the first qubit by varying the factor in the rotation:

OpenQASM 2.0

```
OPENQASM 2.0;
include "qelib1.inc";

qreg q[3];
creg c0[3];
reset q[0];
reset q[1];
reset q[2];
ry(2 * pi / 6) q[0];
barrier q[0], q[1], q[2];
h q[1];
cx q[1], q[2];
barrier q[0], q[1], q[2];
cx q[0], q[1];
h q[0];
barrier q[0], q[1], q[2];
measure q[0] -> c0[0];
measure q[1] -> c0[1];

```

Change this from Quiskit (read only) to OpenQASM, then you can change the program.

Modify the rotation