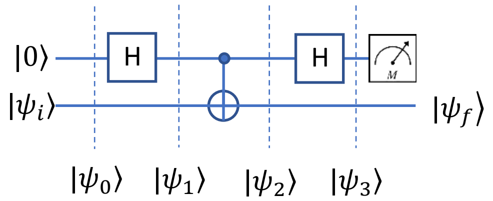
Calculating quantum gates can be done on a conceptual level or basic level. We show this with the example of a combination of Hadamard and CNOT gates.

Example

We use the following circuit:



The meter sign denotes measurement in $\{|0\rangle, |1\rangle\}$ basis.

Suppose $|\psi_i\rangle = a|0\rangle + b|1\rangle$, $a^2 + b^2 = 1$.

Conceptual level

We rewrite the initial state as:

$$|\psi_0\rangle = |0\rangle(a|0\rangle + b|1\rangle) = |0\rangle a|0\rangle + |0\rangle b|1\rangle = a|00\rangle + b|01\rangle$$

Note: constants can float freely through products.

Note: $|0\rangle|0\rangle = |00\rangle = |0\rangle \otimes 0\rangle$ represents the tensor product or Kronecker product.

1) We apply the first Hadamard to $|\psi_0\rangle$ and get $|\psi_1\rangle$:

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)(a|0\rangle + b|1\rangle) =$$

$$\frac{1}{\sqrt{2}}(a|00\rangle + b|01\rangle + a|10\rangle + b|11\rangle)$$

Note: The Hadamard

- acts on the first qubit only
- changes $|0\rangle$ to $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.
- changes $|1\rangle$ to $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$.

2) We apply the CNOT onto $|\psi_1\rangle$. The CNOT swaps the second qubit if the first qubit is $|1\rangle$. In case the first qubit is $|0\rangle$ the CNOT performs no action.

We apply the CNOT:

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}(a|00\rangle + b|01\rangle + a|11\rangle + b|10\rangle) = \frac{1}{\sqrt{2}}(a|00\rangle + b|01\rangle + b|10\rangle + a|11\rangle)$$

3) We apply the second Hadamard to $|\psi_2\rangle$. The Hadamard acts on the first bit only.

We separate the first qubit from $|\psi_2\rangle$:

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}|0\rangle(a|0\rangle + b|1\rangle) + \frac{1}{\sqrt{2}}|1\rangle(b|0\rangle + a|1\rangle)$$

We apply the Hadamard:

$$|\psi_3\rangle = \frac{1}{2}(|0\rangle + |1\rangle)(a|0\rangle + b|1\rangle) + \frac{1}{2}(|0\rangle - |1\rangle)(b|0\rangle + a|1\rangle)$$

We calculate:

$$\frac{1}{2}(|0\rangle + |1\rangle)(a|0\rangle + b|1\rangle) + \frac{1}{2}(|0\rangle - |1\rangle)(b|0\rangle + a|1\rangle) =$$

$$\frac{1}{2}(a|00\rangle + b|01\rangle + a|10\rangle + b|11\rangle + b|00\rangle + a|01\rangle - b|10\rangle - a|11\rangle) =$$

$$\frac{1}{2}\big((a+b)|00\rangle + (b+a)|01\rangle + (a-b)|10\rangle + (b-a)|11\rangle\big) =$$

$$\frac{1}{2}\big((a+b)|00\rangle + (a+b)|01\rangle + (a-b)|10\rangle - (a-b)|11\rangle\big)$$

We can write this as:

$$\frac{1}{2}\big((a+b)(|00\rangle + |01\rangle) + (a-b)(|10\rangle - |11\rangle)\big)$$

Remark: This shows that applying the CNOT between the two Hadamard changes the first qubit, even though the CNOT only affects the second qubit.

What we want to show now is the same calculation done on the basic level with the state vectors.

Basic level

We use the Kronecker product and write all states explicitly.

We begin with two qubits:

$$|\psi_0\rangle = |0\rangle(a|0\rangle + b|1\rangle)$$

We translate into state vectors:

$$|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}, |1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$$

We get:

$$|\psi_0\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \left(a\begin{pmatrix}1\\0\end{pmatrix} + b\begin{pmatrix}0\\1\end{pmatrix}\right) = \begin{pmatrix}1\\0\end{pmatrix} \otimes \left(\begin{pmatrix}a\\0\end{pmatrix} + \begin{pmatrix}0\\b\end{pmatrix}\right) = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}a\\b\end{pmatrix}$$

We build the Kronecker product:

$$\begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}a\\b\end{pmatrix} = \begin{matrix}1 \cdot \begin{pmatrix}a\\b\end{pmatrix}\\ 0 \cdot \begin{pmatrix}a\\b\end{pmatrix}\end{matrix} = \begin{pmatrix}a\\b\\0\\0\end{pmatrix}$$

We apply the first Hadamard. The Hadamard is built of the Kronecker product $H \otimes I$.

We build the Hadamard matrix:

$$H \otimes I = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix} \otimes \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = \frac{1}{\sqrt{2}}\left(\begin{array}{cc|cc}1 & 0 & 1 & 0\\0 & 1 & 0 & 1\\ \hline 1 & 0 & -1 & 0\\0 & 1 & 0 & -1\end{array}\right)$$

We apply the Hadamard:

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\\ 1 & 0 & -1 & 0\\ 0 & 1 & 0 & -1\end{pmatrix}\begin{pmatrix}a\\ b\\ 0\\ 0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}a\\ b\\ a\\ b\end{pmatrix} \quad ①$$

We apply the CNOT. We use the CNOT-matrix:

$$\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{pmatrix}$$

<table>
<tr><td colspan="2">We check:</td></tr>
<tr><td>$\begin{pmatrix}1\\ 0\end{pmatrix}\otimes\begin{pmatrix}a\\ b\end{pmatrix} = \begin{pmatrix}a\\ b\\ 0\\ 0\end{pmatrix}$
$\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{pmatrix}\begin{pmatrix}a\\ b\\ 0\\ 0\end{pmatrix} = \begin{pmatrix}a\\ b\\ 0\\ 0\end{pmatrix} = \begin{pmatrix}1\\ 0\end{pmatrix}\otimes\begin{pmatrix}a\\ b\end{pmatrix}$</td><td>$\begin{pmatrix}0\\ 1\end{pmatrix}\otimes\begin{pmatrix}a\\ b\end{pmatrix} = \begin{pmatrix}0\\ 0\\ a\\ b\end{pmatrix}$
$\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{pmatrix}\begin{pmatrix}0\\ 0\\ a\\ b\end{pmatrix} = \begin{pmatrix}0\\ 0\\ b\\ a\end{pmatrix} = \begin{pmatrix}0\\ 1\end{pmatrix}\otimes\begin{pmatrix}b\\ a\end{pmatrix}$</td></tr>
<tr><td>This works as expected.</td><td>This changes the second qubit as needed.</td></tr>
</table>

We apply the CNOT to $|\psi_1\rangle$:

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{pmatrix}\begin{pmatrix}a\\ b\\ a\\ b\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}a\\ b\\ b\\ a\end{pmatrix} = |\psi_2\rangle \quad ②$$

We apply the Hadamard a second time:

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\\ 1 & 0 & -1 & 0\\ 0 & 1 & 0 & -1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}a\\ b\\ b\\ a\end{pmatrix} =$$

$$\frac{1}{2}\begin{pmatrix}a+b\\ b+a\\ a-b\\ b-a\end{pmatrix} = \frac{1}{2}\begin{pmatrix}a+b\\ a+b\\ a-b\\ -(a-b)\end{pmatrix} = |\psi_3\rangle \quad ③$$

We compare with the high-level solution:

$$|\psi_3\rangle = \frac{1}{2}\big((a+b)(|00\rangle + |01\rangle) + (a-b)(|10\rangle - |11\rangle)\big)$$

We resolve the four 4D basis vectors:

$|00\rangle = |0\rangle|0\rangle = |0\rangle\otimes|0\rangle = \begin{pmatrix}1\\ 0\end{pmatrix}\otimes\begin{pmatrix}1\\ 0\end{pmatrix} = \begin{pmatrix}1\\ 0\\ 0\\ 0\end{pmatrix}$	$|01\rangle = |0\rangle|1\rangle = |0\rangle\otimes|1\rangle = \begin{pmatrix}1\\ 0\end{pmatrix}\otimes\begin{pmatrix}0\\ 1\end{pmatrix} = \begin{pmatrix}0\\ 1\\ 0\\ 0\end{pmatrix}$

$\|10\rangle = \|1\rangle\|0\rangle = \|1\rangle \otimes \|0\rangle = \binom{0}{1} \otimes \binom{1}{0} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$	$\|11\rangle = \|1\rangle\|1\rangle = \|1\rangle \otimes \|1\rangle = \binom{0}{1} \otimes \binom{0}{1} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

We get:

$$|\psi_3\rangle = \frac{1}{2}\left((a+b)\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + (a-b)\begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \right) = \frac{1}{2}\begin{pmatrix} a+b \\ a+b \\ a-b \\ -(a-b) \end{pmatrix}$$

Both methods yield the same result.

Result

Classically two bits are independent. Manipulating one bit doesn't change the other bits.

qubits are not independent. The two qubits from our example build a superposition, a four-dimensional vector.

The initial state $\psi_i\rangle$ is composed of the qubit $|0\rangle$ and the state $|\psi_i\rangle = a|0\rangle + b|1\rangle$. This can be calculated via the Kronecker product:

$$|0\rangle \otimes (a|0\rangle + b|1\rangle)$$

Written as vectors:

$$\binom{1}{0} \otimes \binom{a}{b} = \begin{pmatrix} a \\ b \\ 0 \\ 0 \end{pmatrix}; |\psi_0\rangle$$

Note: This is the superposition 4-D-representation of our 2 by 2 initial state.

Now, following the calculation in the basic level we see that the Hadamard changes the 4-D-representation ①:

$$(H \otimes I)\begin{pmatrix} a \\ b \\ 0 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} a \\ b \\ a \\ b \end{pmatrix}; |\psi_1\rangle$$

Applying the Hadamard again would reverse this effect:

$$(H \otimes I)\frac{1}{\sqrt{2}}\begin{pmatrix} a \\ b \\ a \\ b \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2a \\ 2b \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ b \\ 0 \\ 0 \end{pmatrix}$$

But the CNOT ② interferes and reverses line three and four to:

$$\begin{pmatrix} a \\ b \\ b \\ a \end{pmatrix}; |\psi_2\rangle$$

The Hadamard, applied in ③ picks up the interference and does not produce	but instead:
$\begin{pmatrix} a \\ b \\ 0 \\ 0 \end{pmatrix}$	$\frac{1}{2}\begin{pmatrix} a+b \\ b+a \\ a-b \\ -(a-b) \end{pmatrix}; \lvert\psi_3\rangle$

The effect: The first qubit changes.

Probabilities

We examine the probability for measuring $|0\rangle$ for the first qubit at the end of the calculation.

Conceptual level

We have:

$$\frac{1}{2}\big((a+b)(|00\rangle+|01\rangle)+(a-b)(|10\rangle-|11\rangle)\big)$$

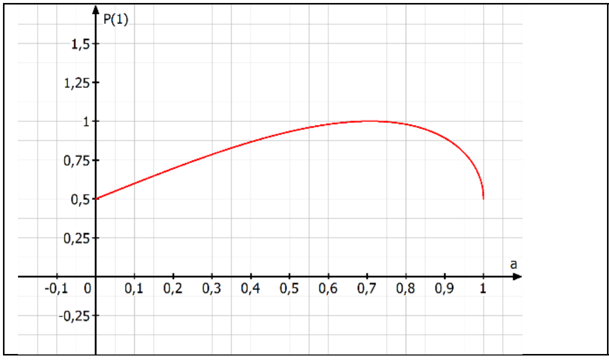
We separate the first qubit:

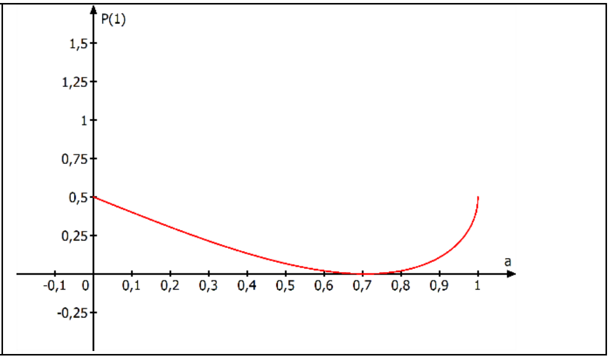
$$\frac{1}{2}\big((a+b)(|00\rangle+|01\rangle)+(a-b)(|10\rangle-|11\rangle)\big)=$$

$$\frac{1}{2}\big(|0\rangle(a+b)(|0\rangle+|1\rangle)+|1\rangle(a-b)(|0\rangle-|1\rangle)\big)=$$

$$|0\rangle\left(\frac{a+b}{2}|0\rangle+\frac{a+b}{2}|1\rangle\right)+|1\rangle\left(\frac{a-b}{2}|0\rangle-\frac{a-b}{2}|1\rangle\right)$$

We calculate the probability for $\lvert 0\rangle$: $\left(\frac{a+b}{2}\right)^2+\left(\frac{a+b}{2}\right)^2=\frac{(a+b)^2}{4}+\frac{(a+b)^2}{4}=$	We calculate the probability for $\lvert 1\rangle$: $\left(\frac{a-b}{2}\right)^2+\left(\frac{a-b}{2}\right)^2=\frac{(a-b)^2}{4}+\frac{(a-b)^2}{4}=$
$\frac{1}{2}(a+b)^2=\frac{a^2+2ab+b^2}{2}$	$\frac{1}{2}(a-b)^2=\frac{a^2-2ab+b^2}{2}$
We use that $a^2+b^2=1$: $a^2+b^2=1\rightarrow b^2=1-a^2\rightarrow b=\sqrt{1-a^2}$ We get:	
$\frac{a^2+2ab+b^2}{2}\rightarrow$ $\frac{1+2ab}{2}=$ $\frac{1+2a\sqrt{1-a^2}}{2}$	$\frac{a^2-2ab+b^2}{2}\rightarrow$ $\frac{1-2ab}{2}=$ $\frac{1-2a\sqrt{1-a^2}}{2}$
This is the probability to measure $\lvert 0\rangle$ for the first qubit depending on a:	This is the probability to measure $\lvert 1\rangle$ for the first qubit depending on a:





Basic level:

We have the 4D state vector:

$$\frac{1}{2}\begin{pmatrix} a+b \\ b+a \\ a-b \\ -(a-b) \end{pmatrix}$$

We remember:

$$\frac{1}{2}\begin{pmatrix} a+b \\ b+a \\ a-b \\ -(a-b) \end{pmatrix} \begin{matrix} \rightarrow |00\rangle \\ \rightarrow |01\rangle \\ \rightarrow |10\rangle \\ \rightarrow |11\rangle \end{matrix}$$

Measuring the first qubit is the projection of the state vector onto one of the two basis vectors:

$$\frac{1}{2}\begin{pmatrix} a+b \\ a+b \end{pmatrix} \leftarrow \frac{1}{2}\begin{pmatrix} a+b \\ b+a \\ a-b \\ -(a-b) \end{pmatrix} \rightarrow \frac{1}{2}\begin{pmatrix} a-b \\ -(a-b) \end{pmatrix}$$

$\frac{1}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} a+b \\ a+b \end{pmatrix} = \frac{1}{2}\begin{pmatrix} a+b \\ a+b \\ 0 \\ 0 \end{pmatrix}$	$\frac{1}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} a-b \\ -(a-b) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 \\ 0 \\ a-b \\ -(a-b) \end{pmatrix}$

We calculate the probability for measuring $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$: $\frac{1}{2}\begin{pmatrix} a+b \\ a+b \end{pmatrix} \rightarrow$ $\frac{1}{4}(a+b)^2 + (a+b)^2 =$ $\frac{a^2+2ab+b^2}{2} = \frac{1+2ab}{2}$	We calculate the probability for measuring $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$: $\frac{1}{2}\begin{pmatrix} a-b \\ -(a-b) \end{pmatrix} \rightarrow$ $\frac{1}{2}\sqrt{(a-b)^2 + \left(-(a-b)\right)^2} = \frac{1-2ab}{2}$

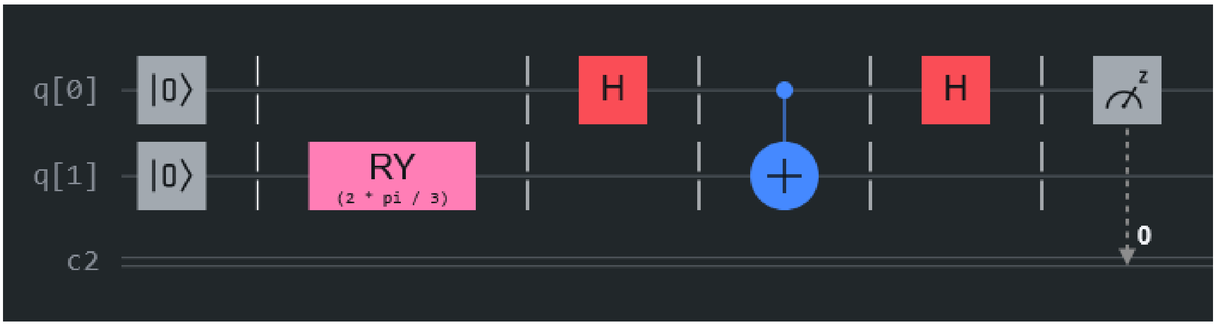
This is the same probability we got at the conceptual level. We check this by help of the IBM composer.

You can access the IBM composer via:

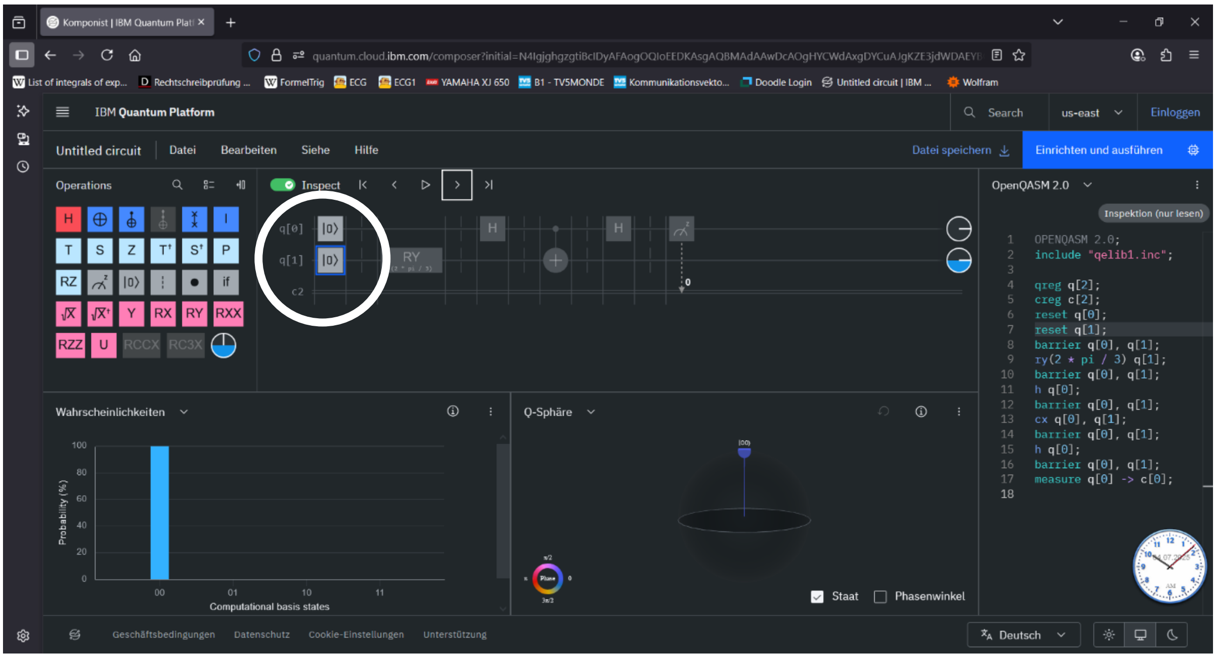
https://quantum.cloud.ibm.com/composer?initial=N4IgjghgzgtiBcIDyAFAogOQIoEEDKAsgAQBMAdAAwDcAOgHYCWdAxgDYCuAJgKZE3jdWDAEYBGMk2b9ademABO3AOZEwAbQAsAXRnNFK5pp316IADQg6EGNwQgAqnQAuDJ626cizBvObtXIAC%2BQA

Note: link checked 2025/07/05

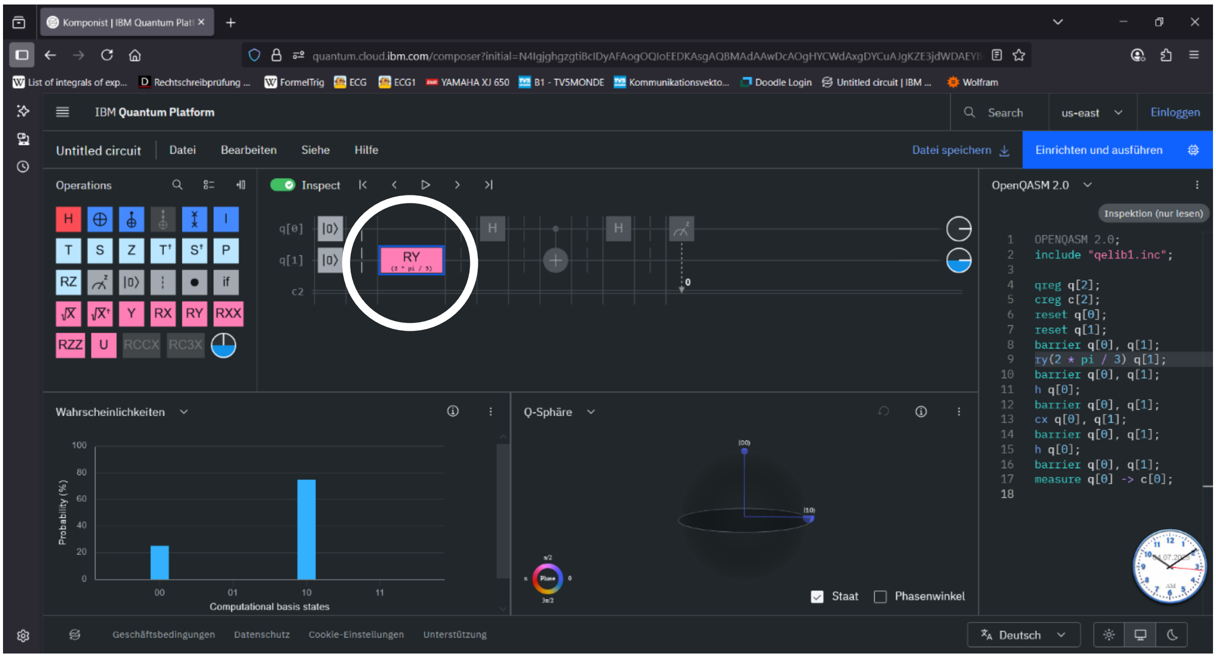
We adjust the values for the probability densities a and b to 0.5 and 0.866 to get probabilities of 0.25 and 0.75. We do this by a rotation of $\pi/3$. Note that in the IBM composer we write $2 \cdot \pi/3$ due to the double angle behavior.



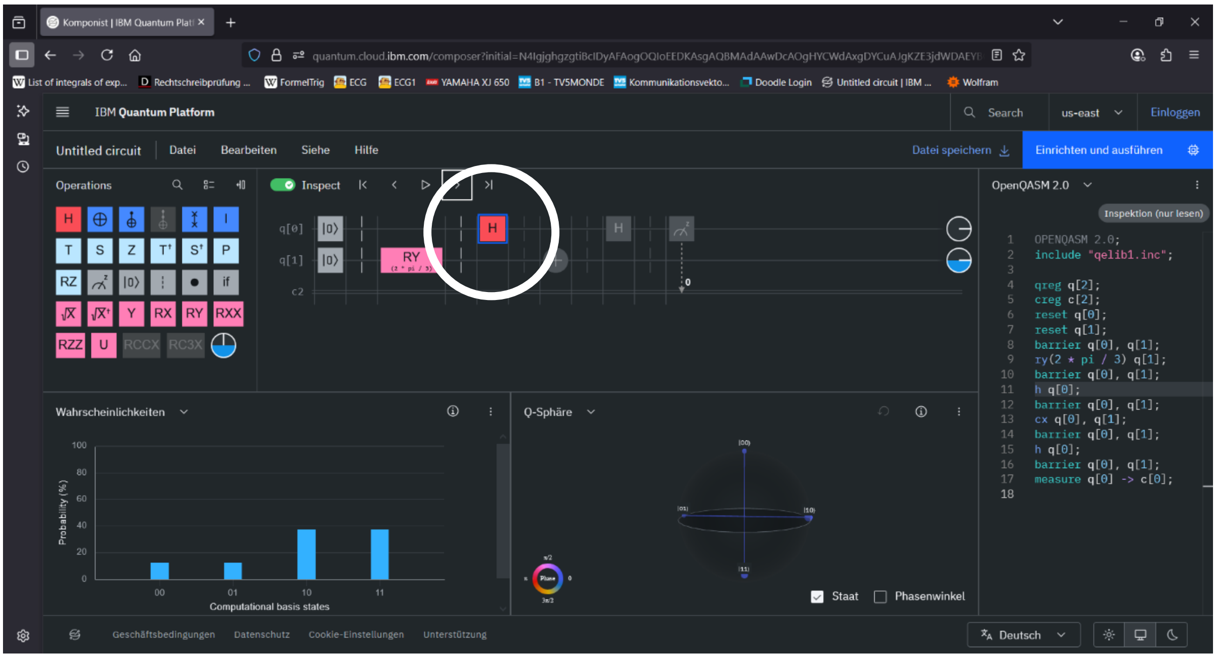
We inspect the circuit step by step.



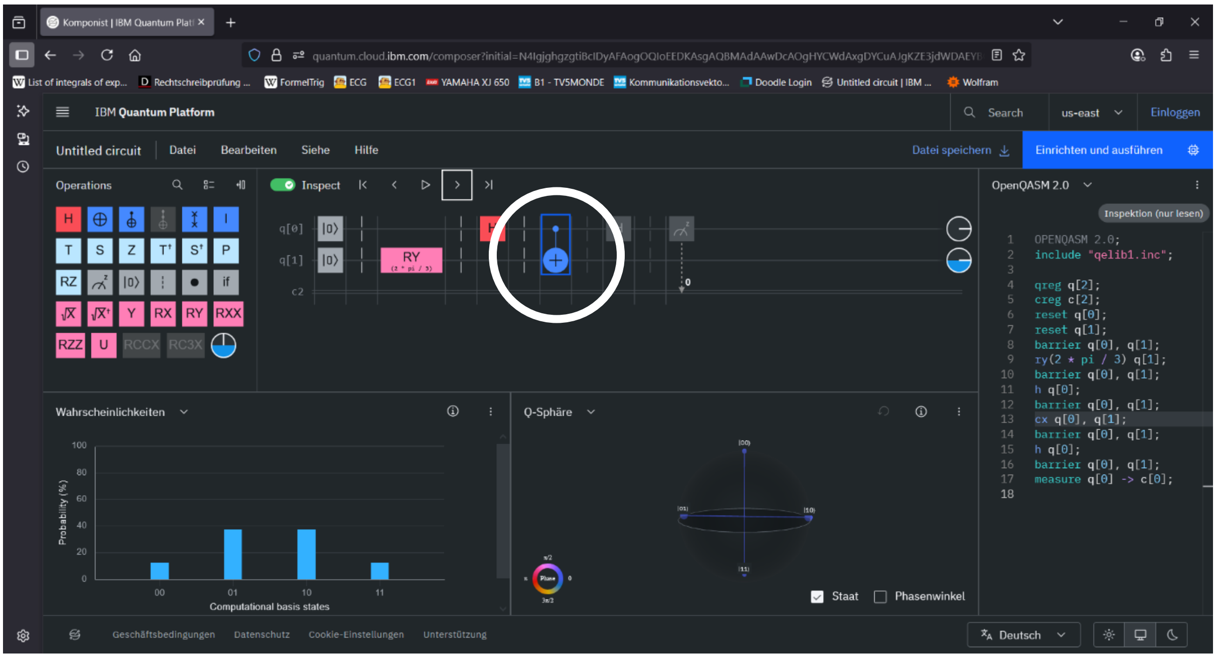
We reset the two qubits to $|00\rangle$. The probability for $|00\rangle$ is 100%.



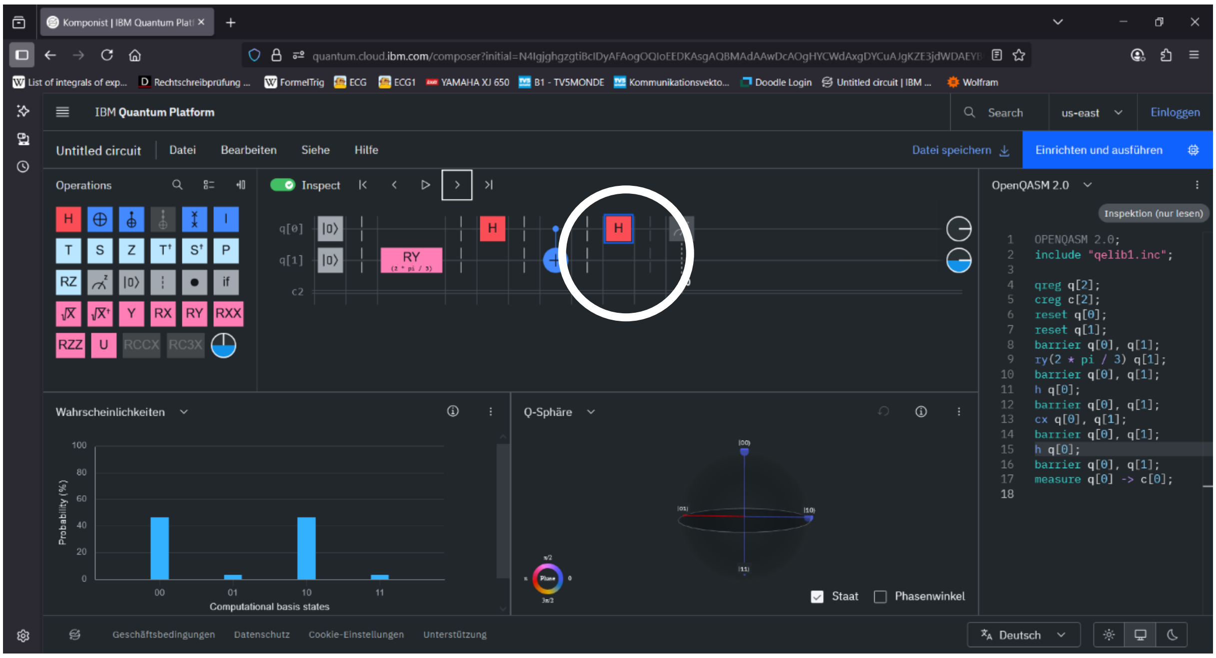
We rotate the second qubit by $\pi/3$ to get the probability densities of 0.5 and 0.866 ($\sqrt{0.75}$) and accordingly the probabilities of 0.25 and 0.75. Note that the order of the qubits in the bottom line is reversed according to the IBM little endian notation. The first qubit in the composer line is the second qubit in the bottom line.



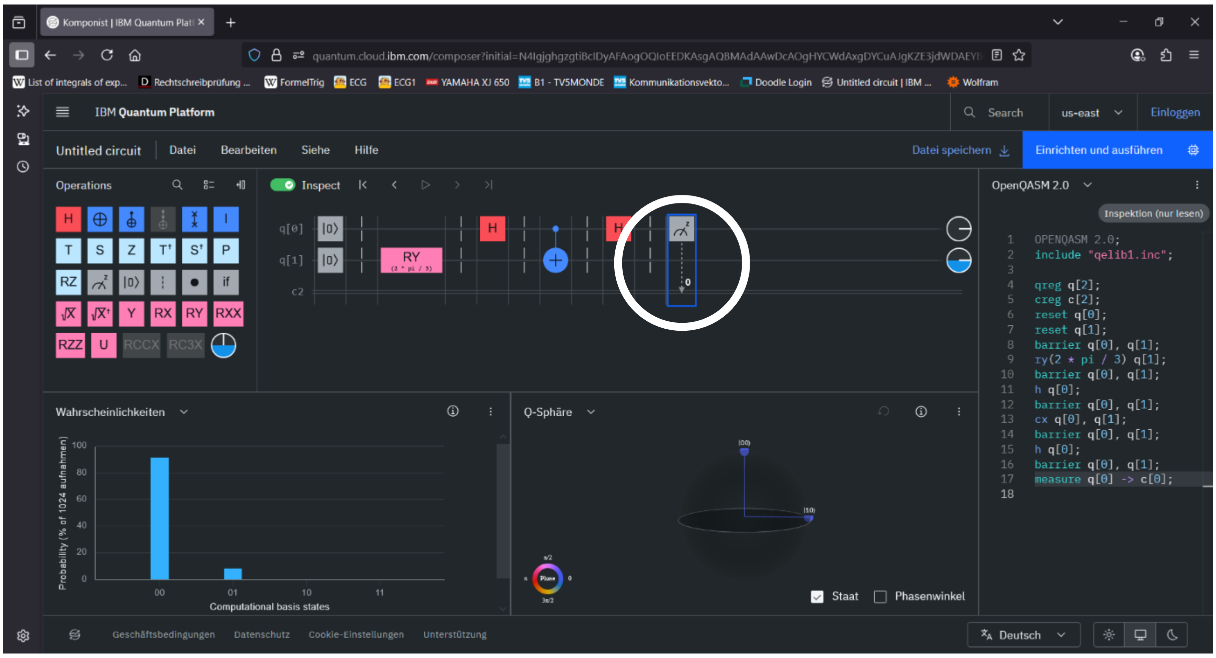
We applied the Hadamard to the first qubit. The probabilities of all qubits have changed. We note the symmetry between 00,01 and 10,11.



We applied the CNOT to the first qubit. Again, the probabilities of all qubits change. We note the symmetry between 00, 11 and 01, 10.



We applied the second Hadamard to the first qubit. Again, the probabilities of all qubits change. We note the symmetry between 00, 10 and 01, 11.



Measuring the first qubit destroys it. The picture shows the measuring of the first qubit (the right one in IBM notation) with probability of about 95% to zero and 5% to one.

The second qubit (the left one in IBM notation) is always zero, |0⟩.

Both qubits changed their values.