

This paper deals with a general Galilean invariance of the Schroedinger equation.

It is part of an online course at MIT you find at: <https://courses.learn.mit.edu/learn/course/course-v1:MITxT+8.04x+1T2026/home>

Hope I can help you learning quantum mechanics.

We use the free particle 1D Schroedinger equation for the wavefunction $\Psi(x, t)$:

$$i\hbar \frac{\partial \Psi}{\partial x} = -\frac{\hbar^2}{2m} \cdot \frac{\partial^2}{\partial x^2} \Psi(x, t) \rightarrow \frac{\partial \Psi}{\partial x} = i \frac{\hbar}{2m} \cdot \frac{\partial^2}{\partial x^2} \Psi(x, t)$$

The Schroedinger equation is independent under Galilean transformation:

$$x' = x - v \cdot t, t' = t$$

This means that there is a wavefunction $\Psi'(x', t') = f(x, t) \cdot \Psi(x, t)$ with the following properties:

1. $f(x, t)$ independent of $\Psi(x, t)$
because it should be valid for all $\Psi(x, t)$.
2. $|f(x, t)| = 1$ giving $|\Psi'(x', t')| = |\Psi(x, t)|$
if $f(x, t)$ is a real function this would mean $f(x, t) = \pm 1$. If $f(x, t)$ is a complex function this would mean it is a phase only.
3. $f(x = 0, t = 0) = 1$
if both inertial systems match the wave functions are identic: $\Psi'(x', t') = \Psi(x, t)$

Under a Galilean transformation we have derivatives in the primed coordinate system:

$$\frac{\partial}{\partial x'} = \frac{\partial}{\partial x}, \quad \frac{\partial}{\partial t'} = v \frac{\partial}{\partial x} + \frac{\partial}{\partial t}$$

The wave function in the Galilean transformed coordinate system must obey:

$$\frac{\partial'}{\partial t'} \Psi'(x', t') = i \frac{\hbar}{2m} \cdot \frac{\partial^2}{\partial x'^2} \Psi'(x', t')$$

We use $\frac{\partial}{\partial t'} = v \frac{\partial}{\partial x} + \frac{\partial}{\partial t}$ and $\Psi'(x', t') = f(x, t) \cdot \Psi(x, t)$.

Left side:

$$\begin{aligned} & \left(v \frac{\partial}{\partial x} + \frac{\partial}{\partial t} \right) f(x, t) \cdot \Psi(x, t) = \\ & v \left(\frac{\partial}{\partial x} f(x, t) \right) \Psi(x, t) + v f(x, t) \frac{\partial}{\partial x} \Psi(x, t) + \left(\frac{\partial}{\partial t} f(x, t) \right) \Psi(x, t) + f(x, t) \left(\frac{\partial}{\partial t} \Psi(x, t) \right) \end{aligned}$$

Right side:

$$\begin{aligned} & i \frac{\hbar}{2m} \cdot \frac{\partial^2}{\partial x'^2} \Psi'(x', t') = i \frac{\hbar}{2m} \cdot \frac{\partial}{\partial x} \left(\frac{\partial}{\partial x} \Psi'(x', t') \right) = \\ & i \frac{\hbar}{2m} \cdot \frac{\partial}{\partial x} \left(\left(\frac{\partial}{\partial x} f(x, t) \right) \cdot \Psi(x, t) + f(x, t) \frac{\partial}{\partial x} \Psi(x, t) \right) = \\ & i \frac{\hbar}{2m} \left(\left(\frac{\partial^2}{\partial x^2} f(x, t) \right) \Psi(x, t) + \left(\frac{\partial}{\partial x} f(x, t) \right) \left(\frac{\partial}{\partial x} \Psi(x, t) \right) + \left(\frac{\partial}{\partial x} f(x, t) \right) \frac{\partial}{\partial x} \Psi(x, t) + f(x, t) \frac{\partial^2}{\partial x^2} \Psi(x, t) \right) = \\ & i \frac{\hbar}{2m} \left(\left(\frac{\partial^2}{\partial x^2} f(x, t) \right) \Psi(x, t) + 2 \left(\frac{\partial}{\partial x} f(x, t) \right) \left(\frac{\partial}{\partial x} \Psi(x, t) \right) + f(x, t) \frac{\partial^2}{\partial x^2} \Psi(x, t) \right) \end{aligned}$$

We get:

$$v \left(\frac{\partial}{\partial x} f(x, t) \right) \Psi(x, t) + v f(x, t) \frac{\partial}{\partial x} \Psi(x, t) + \left(\frac{\partial}{\partial t} f(x, t) \right) \Psi(x, t) + f(x, t) \left(\frac{\partial}{\partial t} \Psi(x, t) \right) =$$

$$i \frac{\hbar}{2m} \left(\left(\frac{\partial^2}{\partial x^2} f(x, t) \right) \Psi(x, t) + 2 \left(\frac{\partial}{\partial x} f(x, t) \right) \left(\frac{\partial}{\partial x} \Psi(x, t) \right) + f(x, t) \frac{\partial^2}{\partial x^2} \Psi(x, t) \right)$$

The last expression on the left and the right side fulfill the Schrodinger equation and can be cancelled.

We get:

$$v \left(\frac{\partial}{\partial x} f(x, t) \right) \Psi(x, t) + v f(x, t) \frac{\partial}{\partial x} \Psi(x, t) + \left(\frac{\partial}{\partial t} f(x, t) \right) \Psi(x, t) =$$

$$i \frac{\hbar}{2m} \left(\left(\frac{\partial^2}{\partial x^2} f(x, t) \right) \Psi(x, t) + 2 \left(\frac{\partial}{\partial x} f(x, t) \right) \left(\frac{\partial}{\partial x} \Psi(x, t) \right) \right)$$

We transform:

$$v \left(\frac{\partial}{\partial x} f(x, t) \right) \Psi(x, t) + v f(x, t) \frac{\partial}{\partial x} \Psi(x, t) + \left(\frac{\partial}{\partial t} f(x, t) \right) \Psi(x, t) =$$

$$i \frac{\hbar}{2m} \left(\frac{\partial^2}{\partial x^2} f(x, t) \right) \Psi(x, t) + i \frac{\hbar}{m} \left(\frac{\partial}{\partial x} f(x, t) \right) \left(\frac{\partial}{\partial x} \Psi(x, t) \right)$$

$$v \left(\frac{\partial}{\partial x} f(x, t) \right) \Psi(x, t) + v f(x, t) \frac{\partial}{\partial x} \Psi(x, t) + \left(\frac{\partial}{\partial t} f(x, t) \right) \Psi(x, t) - i \frac{\hbar}{2m} \left(\frac{\partial^2}{\partial x^2} f(x, t) \right) \Psi(x, t)$$

$$- i \frac{\hbar}{m} \left(\frac{\partial}{\partial x} f(x, t) \right) \left(\frac{\partial}{\partial x} \Psi(x, t) \right) = 0$$

$$v \left(\frac{\partial}{\partial x} f(x, t) \right) \Psi(x, t) + \left(\frac{\partial}{\partial t} f(x, t) \right) \Psi(x, t) - i \frac{\hbar}{2m} \left(\frac{\partial^2}{\partial x^2} f(x, t) \right) \Psi(x, t)$$

$$- i \frac{\hbar}{m} \left(\frac{\partial}{\partial x} f(x, t) \right) \left(\frac{\partial}{\partial x} \Psi(x, t) \right) + v f(x, t) \frac{\partial}{\partial x} \Psi(x, t) = 0$$

$$\left(v \left(\frac{\partial}{\partial x} f(x, t) \right) + \left(\frac{\partial}{\partial t} f(x, t) \right) - i \frac{\hbar}{2m} \left(\frac{\partial^2}{\partial x^2} f(x, t) \right) \right) \Psi(x, t)$$

$$- \left(i \frac{\hbar}{m} \left(\frac{\partial}{\partial x} f(x, t) \right) - v f(x, t) \right) \frac{\partial}{\partial x} \Psi(x, t) = 0$$

The sum is made of an expression depending on $\Psi(x, t)$ and $\frac{\partial}{\partial x} \Psi(x, t)$. To vanish for any $\Psi(x, t)$ both parts must vanish independently:

$$v \left(\frac{\partial}{\partial x} f(x, t) \right) + \left(\frac{\partial}{\partial t} f(x, t) \right) - i \frac{\hbar}{2m} \left(\frac{\partial^2}{\partial x^2} f(x, t) \right) = 0$$

$$i \frac{\hbar}{m} \left(\frac{\partial}{\partial x} f(x, t) \right) - v f(x, t) = 0$$

We rewrite the second equation:

$$\frac{\partial f(x, t)}{\partial x} = -i \frac{mv}{\hbar} f(x, t)$$

We separate the variables:

$$\frac{\partial f(x, t)}{f(x, t)} = -i \frac{mv}{\hbar} \partial x$$

We integrate:

$$\ln(f(x, t)) = -i \frac{mv}{\hbar} x$$

$$f(x, t) = e^{-i \frac{mv}{\hbar} x} \cdot g(t)$$

Note: As we use partial differentiation, $g(t)$ is a function depending on time t only.

Note: As $f(x, t)$ is a phase we need $|f(x, t)| = 1$ forcing the integration constant to be 1.

We get:

$$\frac{\partial f(x, t)}{\partial x} = -i \frac{mv}{\hbar} e^{-i \frac{mv}{\hbar} x} \cdot g(t)$$

We need the second partial derivative:

$$\frac{\partial^2 f(x, t)}{\partial x^2} = \left(-i \frac{mv}{\hbar} \right) \left(-i \frac{mv}{\hbar} \right) e^{-i \frac{mv}{\hbar} x} \cdot g(t) =$$

$$- \left(\frac{mv}{\hbar} \right)^2 e^{-i \frac{mv}{\hbar} x} \cdot g(t)$$

We insert this into the first equation:

$$v \left(\frac{\partial}{\partial x} f(x, t) \right) + \left(\frac{\partial}{\partial t} f(x, t) \right) - i \frac{\hbar}{2m} \left(\frac{\partial^2}{\partial x^2} f(x, t) \right) = 0$$

$$-i v \frac{mv}{\hbar} e^{-i \frac{mv}{\hbar} x} \cdot g(t) + e^{-i \frac{mv}{\hbar} x} \cdot \frac{\partial}{\partial t} g(t) - i \frac{\hbar}{2m} \left(- \left(\frac{mv}{\hbar} \right)^2 e^{-i \frac{mv}{\hbar} x} \cdot g(t) \right) = 0$$

$$-i \frac{mv^2}{\hbar} \cdot g(t) + \frac{\partial}{\partial t} g(t) - i \frac{\hbar}{2m} \left(- \left(\frac{mv}{\hbar} \right)^2 \cdot g(t) \right) = 0$$

$$-i \frac{mv^2}{\hbar} \cdot g(t) + \frac{\partial}{\partial t} g(t) + i \frac{mv^2}{2\hbar} \cdot g(t) = 0$$

$$\frac{\partial}{\partial t} g(t) - i \frac{mv^2}{2\hbar} \cdot g(t) = 0$$

$$\frac{\partial g(t)}{\partial t} = i \frac{mv^2}{2\hbar} \cdot g(t)$$

We separate variables:

$$\frac{\partial g(t)}{g(t)} = i \frac{mv^2}{2\hbar} \partial t$$

We integrate:

$$\ln(g(t)) + c = i \frac{mv^2}{2\hbar} t + c$$

$$g(t) = e^{i \frac{mv^2}{2\hbar} t} \cdot C$$

Note: As $f(x, t)$ is a phase we need $|C| = 1$ forcing the integration constant C to be a phase:

$$C = e^{i\theta}$$

The property $f(x = 0, t = 0) = 1$ forces the global phase $\theta = 0$.

We combine:

$$f(x, t) = e^{-i \frac{mv}{\hbar} x} \cdot e^{i \frac{mv^2}{2\hbar} t} = e^{i \left(-\frac{mv}{\hbar} x + \frac{mv^2}{2\hbar} t \right)}$$

Result

Under the Galilean transformation the wave function changes:

$$\Psi'(x', t') = f(x, t) \cdot \Psi(x, t) = e^{i \left(-\frac{mv}{\hbar} x + \frac{mv^2}{2\hbar} t \right)} \cdot \Psi(x, t)$$

Application

The wave function a non-relativistic particle:

$$\Psi(x, t) = A e^{i(kx - \omega t)}$$

Using $p = \hbar k$ and $E = \hbar \omega = \frac{p^2}{2m}$ we write:

$$\Psi(x, t) = A e^{i \left(\frac{p}{\hbar} x - \frac{p^2}{2\hbar m} t \right)}$$

We use the transformation:

$$\begin{aligned} \Psi'(x', t') &= e^{i \left(-\frac{mv}{\hbar} x + \frac{mv^2}{2\hbar} t \right)} \cdot \Psi(x, t) = \\ &= e^{i \left(-\frac{mv}{\hbar} x + \frac{mv^2}{2\hbar} t \right)} \cdot A e^{i \left(\frac{p}{\hbar} x - \frac{p^2}{2\hbar m} t \right)} = \\ &= A e^{i \left(\left(\frac{p}{\hbar} - \frac{mv}{\hbar} \right) x - \left(\frac{p^2}{2\hbar m} - \frac{mv^2}{2\hbar} \right) t \right)} \end{aligned}$$

We write the wave function in the coordinates of the moving inertial system:

$$x = x' + vt', t = t'$$

We get:

$$\begin{aligned}
 \Psi'(x', t') &= Ae^{i\left(\left(\frac{p}{\hbar} - \frac{mv}{\hbar}\right)x - \left(\frac{p^2}{2\hbar m} - \frac{mv^2}{2\hbar}\right)t\right)} = \\
 &Ae^{i\left(\left(\frac{p}{\hbar} - \frac{mv}{\hbar}\right)(x' + vt') - \left(\frac{p^2}{2\hbar m} - \frac{mv^2}{2\hbar}\right)t'\right)} = \\
 &Ae^{i\left(\left(\frac{p}{\hbar} - \frac{mv}{\hbar}\right)x' + \left(\frac{p}{\hbar} - \frac{mv}{\hbar}\right)vt' - \left(\frac{p^2}{2\hbar m} - \frac{mv^2}{2\hbar}\right)t'\right)} = \\
 &Ae^{i\left(\left(\frac{p}{\hbar} - \frac{mv}{\hbar}\right)x' + \left(\frac{pv}{\hbar} - \frac{mv^2}{\hbar} - \frac{p^2}{2\hbar m} + \frac{mv^2}{2\hbar}\right)t'\right)} = \\
 &Ae^{i\left(\left(\frac{p}{\hbar} - \frac{mv}{\hbar}\right)x' + \left(\frac{pv}{\hbar} - \frac{mv^2}{2\hbar} - \frac{p^2}{2\hbar m}\right)t'\right)} = \\
 &Ae^{i\left(\left(\frac{p}{\hbar} - \frac{mv}{\hbar}\right)x' + \left(\frac{2pvm}{2m\hbar} - \frac{m^2v^2}{2m\hbar} - \frac{p^2}{2\hbar m}\right)t'\right)} = \\
 &Ae^{i\left(\left(\frac{p}{\hbar} - \frac{mv}{\hbar}\right)x' - \frac{1}{2m\hbar}(m^2v^2 - 2pvm + p^2)t'\right)} = \\
 &Ae^{i\left(\left(\frac{p-mv}{\hbar}\right)x' + \frac{(p-mv)^2}{2m\hbar}t'\right)}
 \end{aligned}$$

Result:

$\Psi(x, t) = Ae^{i\left(\frac{p}{\hbar}x - \frac{p^2}{2\hbar m}t\right)}$	$\Psi'(x', t') = Ae^{i\left(\left(\frac{p-mv}{\hbar}\right)x' + \frac{(p-mv)^2}{2m\hbar}t'\right)}$
Momentum p	Momentum $p' = p - mv$
Energy $E = \frac{p^2}{2m}$	Energy $E' = \frac{(p-mv)^2}{2m}$

The relative speed v between the two inertial systems goes directly into momentum and energy.