This paper deals with the free particle in quantum mechanics. It follows

https://courses.learn.mit.edu/learn/course/course-v1:MITxT+8.04x+1T2026/home

Hope I can help you learning quantum mechanics.

Note: This is an experimental text, especially towards the end and may contain errors …

We have a normalized wave packet representing the state of a free particle with mass m at time $t = 0$. At this time the width of the wave packet is a.

$$\psi_a(x,0) = \frac{1}{(2\pi)^{\frac{1}{4}}\sqrt{a}}\ e^{\left(-\frac{x^2}{4a^2}\right)}$$

Note: This is a real function.

The wave function is properly normalized:

$$\int_{-\infty}^{\infty} \left(\psi_a(x,0)\right)^* \psi_a(x,0)dx =$$

$$\frac{1}{(2\pi)^{\frac{1}{2}}a}\int_{-\infty}^{\infty} e^{\left(-\frac{x^2}{2a^2}\right)}dx =;$$

This integral is of type $\int_{-\infty}^{\infty} e^{-\alpha x^2}dx$. Wikipedia delivers:

$$\int_{-\infty}^{\infty} e^{-\alpha x^2}dx = \sqrt{\frac{\pi}{\alpha}}$$

We have $\alpha = \frac{1}{2a^2}$.

We get:

$$\frac{1}{\sqrt{2\pi}a}\int_{-\infty}^{\infty} e^{\left(-\frac{x^2}{2a^2}\right)}dx = \frac{1}{\sqrt{2\pi}a}\sqrt{\frac{\pi}{\frac{1}{2a^2}}} = \frac{1}{\sqrt{2\pi}\cdot a}\sqrt{2a^2\pi} = 1$$

The wave function $\psi_a(x,0)$ is properly normalized.

We use Fourier transform to get $\tilde{\psi}_a$ in momentum space.

$$\tilde{\psi}_a(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \psi_a(x,0)e^{-ikx}dx =$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{1}{(2\pi)^{\frac{1}{4}}\sqrt{a}}\ e^{\left(-\frac{x^2}{4a^2}\right)}e^{-ikx}dx =$$

$$\frac{1}{(2\pi)^{\frac{3}{4}}\sqrt{a}}\int_{-\infty}^{\infty}\ e^{-\left(\frac{x^2}{4a^2}+ikx\right)}dx =;$$

This integral is of type $\int_{-\infty}^{\infty} e^{-(\alpha x^2+\beta x)}dx$. Wikipedia delivers:

$$\int_{-\infty}^{\infty} e^{-(\alpha x^2+\beta x)}dx = \sqrt{\frac{\pi}{\alpha}}e^{\frac{\beta^2}{4\alpha}}$$

We have $\alpha = \frac{1}{4a^2}, \beta = ik$

We get:

$$\frac{1}{(2\pi)^{\frac{3}{4}}\sqrt{a}}\int_{-\infty}^{\infty}\ e^{-\left(\frac{x^2}{4a^2}+ikx\right)}dx = \frac{1}{(2\pi)^{\frac{3}{4}}\sqrt{a}}\sqrt{\frac{\pi}{\frac{1}{4a^2}}}\,e^{\frac{-k^2}{4\frac{1}{4a^2}}} =$$

$$\frac{2\sqrt{a\pi}}{(2\pi)^{\frac{3}{4}}}e^{-a^2k^2} = \left(\frac{2}{\pi}\right)^{\frac{1}{4}} a^{\frac{1}{2}}e^{-a^2k^2}$$

Result:

$$\tilde{\psi}_a(k) = \left(\frac{2}{\pi}\right)^{\frac{1}{4}} a^{\frac{1}{2}}e^{-a^2k^2}$$

We now can use the inverse Fourier transform and rewrite $\psi_a(x,t)$:

$$\psi_a(x,t) = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} \tilde{\psi}_a(k) \cdot e^{i(kx-\omega(k)t)}dk$$

$$\omega(k) = \frac{\hbar k^2}{2m}$$

This is the energy of the particle.

We calculate:

$$\frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} \tilde{\psi}_a(k) \cdot e^{i(kx-\omega(k)t)}dk =$$

$$\frac{1}{(2\pi)^{\frac{1}{2}}} \cdot \left(\frac{2}{\pi}\right)^{\frac{1}{4}} a^{\frac{1}{2}} \int_{-\infty}^{\infty} e^{-a^2k^2} \cdot e^{i\left(kx-\frac{\hbar k^2}{2m}t\right)}dk =$$

$$2^{-\frac{1}{4}}a^{\frac{1}{2}}\pi^{-\frac{3}{4}} \int_{-\infty}^{\infty} e^{-a^2k^2-i\frac{\hbar k^2}{2m}t+ikx}dk =$$

$$2^{-\frac{1}{4}}a^{\frac{1}{2}}\pi^{-\frac{3}{4}} \int_{-\infty}^{\infty} e^{-k^2\left(a^2+i\frac{\hbar}{2m}t\right)+ikx}dk$$

Again, this is of type $\int_{-\infty}^{\infty} e^{-(ax^2+bx)}dx$. Wikipedia states:

$$\int_{-\infty}^{\infty} e^{-(\alpha x^2+\beta x)}dx = \sqrt{\frac{\pi}{\alpha}}e^{\frac{\beta^2}{4\alpha}}$$

We have $\alpha = a^2 + i\frac{\hbar}{2m}t, \beta = -ix$. We resolve β and get for the integral:

$$\int_{-\infty}^{\infty} e^{-k^2\alpha+kix}dk = \sqrt{\frac{\pi}{\alpha}}e^{\frac{-x^2}{4\alpha}}$$

We get for $\psi_a(x,t)$:

$$2^{-\frac{1}{4}}a^{\frac{1}{2}}\pi^{-\frac{3}{4}}\sqrt{\frac{\pi}{\alpha}}e^{\frac{-x^2}{4\alpha}}$$

At time $t = 0$ we have the probability density:

$$|\psi_a(x,0)|^2 = \frac{1}{(2\pi)^{\frac{1}{2}}a}\, e^{\left(-\frac{x^2}{2a^2}\right)}$$

$$\alpha(t=0) = a^2$$

For $t > 0$ we calculate:

$$|\psi_a(x,t)|^2 = \left(2^{-\frac{1}{4}}a^{\frac{1}{2}}\pi^{-\frac{3}{4}}\sqrt{\frac{\pi}{\alpha^*}}e^{\frac{-x^2}{4\alpha^*}}\right) \cdot \left(2^{-\frac{1}{4}}a^{\frac{1}{2}}\pi^{-\frac{3}{4}}\sqrt{\frac{\pi}{\alpha}}e^{\frac{-x^2}{4\alpha}}\right) =$$

$$\alpha = a^2 + i\frac{\hbar}{2m}t$$

$$\alpha^* = a^2 - i\frac{\hbar}{2m}t$$

$$\left(2^{-\frac{1}{4}}a^{\frac{1}{2}}\pi^{-\frac{3}{4}}\sqrt{\frac{\pi}{\alpha^*}}e^{\frac{-x^2}{4\left(a^2-i\frac{\hbar}{2m}t\right)}}\right)\cdot\left(2^{-\frac{1}{4}}a^{\frac{1}{2}}\pi^{-\frac{3}{4}}\sqrt{\frac{\pi}{\alpha}}e^{\frac{-x^2}{4\left(a^2+i\frac{\hbar}{2m}t\right)}}\right)=$$

$$\frac{a}{\pi^{\frac{1}{2}}2^{\frac{1}{2}}\sqrt{|\alpha|^2}}e^{\frac{1}{4}\cdot\left(\frac{-x^2}{a^2-i\frac{\hbar}{2m}t}+\frac{-x^2}{a^2+i\frac{\hbar}{2m}t}\right)}$$

$$\boxed{|\alpha|^2=a^4+\left(\frac{\hbar}{2m}t\right)^2}$$

The exponent is of type $\frac{d}{x-iy}+\frac{d}{x+iy}$:

$$\frac{d}{x-iy}+\frac{d}{x+iy}=\frac{d(x+iy+x-iy)}{|x+iy|^2}=$$

$$\frac{d\cdot 2Re(x+iy)}{|x+iy|^2}$$

We have:

$$d=-x^2, x-iy\equiv a^2-i\frac{\hbar}{2m}t, x+iy=a^2+i\frac{\hbar}{2m}t$$

$$\boxed{Re\left(a^2+i\frac{\hbar}{2m}t\right)=a^2}$$

The exponent:

$$\frac{d\cdot 2Re(x+iy)}{|x+iy|^2}\to\frac{-x^2\cdot 2a^2}{a^4+\left(\frac{\hbar}{2m}t\right)^2}$$

We get:

$$|\psi_a(x,t)|^2=\frac{a}{\pi^{\frac{1}{2}}2^{\frac{1}{2}}\sqrt{\left(a^4+\left(\frac{\hbar}{2m}t\right)^2\right)}}e^{-\frac{1}{2}\cdot\left(\frac{x^2a^2}{a^4+\left(\frac{\hbar}{2m}\right)^2t^2}\right)}=$$

$$\frac{1}{\sqrt{2\pi\left(a^2+\left(\frac{\hbar}{2ma}t\right)^2\right)}}e^{-\frac{1}{2}\cdot\left(\frac{x^2}{a^2+\left(\frac{\hbar}{2ma}t\right)^2}\right)}$$

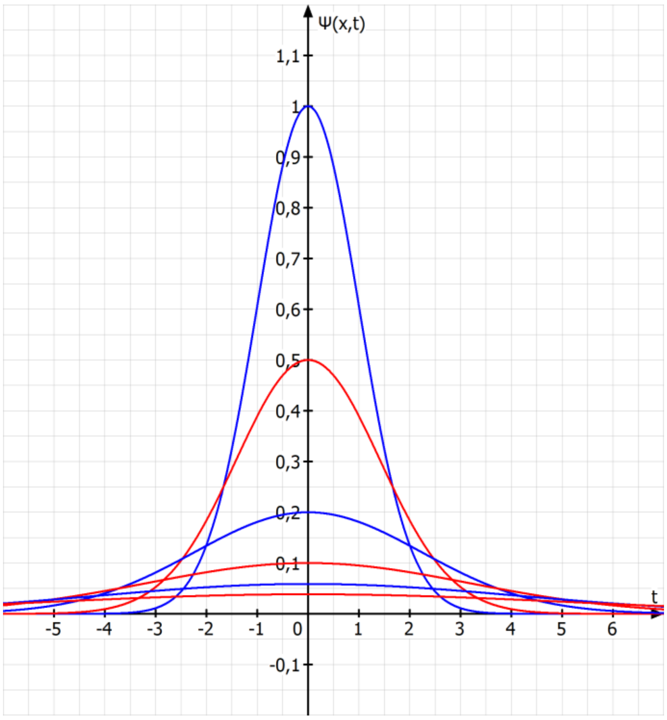
$|\psi_a(x,t)|^2$ is a real function of x and t.

We set all constants to 1 and plot:

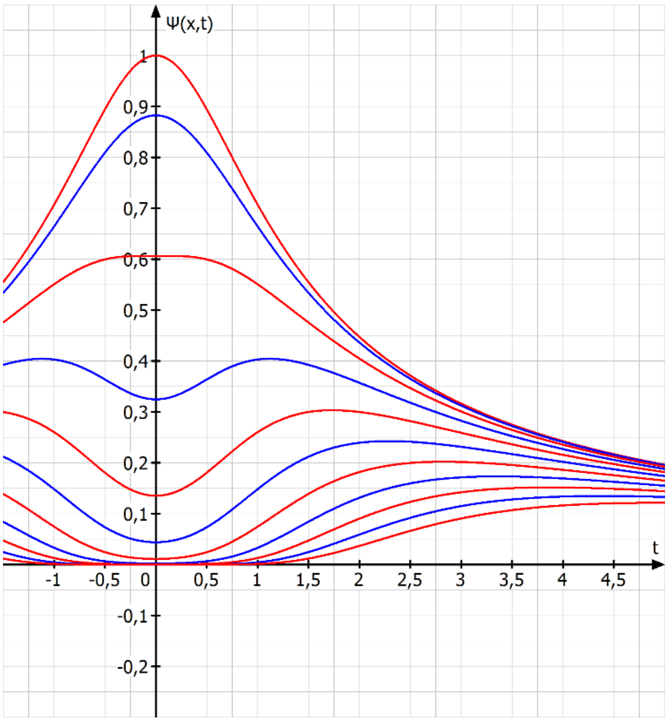
$$|\psi_1(x,t)|^2 = \frac{1}{\sqrt{1+t^2}} e^{-\frac{1}{2}\left(\frac{x^2}{1+t^2}\right)}$$

Each graph represents $|\psi_1(x,t)|^2$ at $|\psi_1(x,t=0)|^2$, $|\psi_1(x,t=1)|^2$, …, $|\psi_1(x,t=5)|^2$.

If we fix time, we get a function of x, a Gaussian each centered at the origin.



If we fix position, we see maxima of the probability density moving from the origin outwards.



Note: x (red/blue) in steps of 0.5 beginning with $x = 0$.

We check:

$$|\psi_1(x,t)|^2 = \frac{1}{\sqrt{1+t^2}} e^{-\frac{1}{2}\left(\frac{x^2}{1+t^2}\right)}$$

$$\frac{\partial}{\partial t}\left(\frac{1}{\sqrt{1+t^2}} e^{-\frac{1}{2}\left(\frac{x^2}{1+t^2}\right)}\right) = -\frac{t \cdot e^{-\frac{0.5*x^2}{t^2+1}}(1-x^2+t^2)}{(t^2+1)^{\frac{5}{2}}}$$

The derivative must be zero:

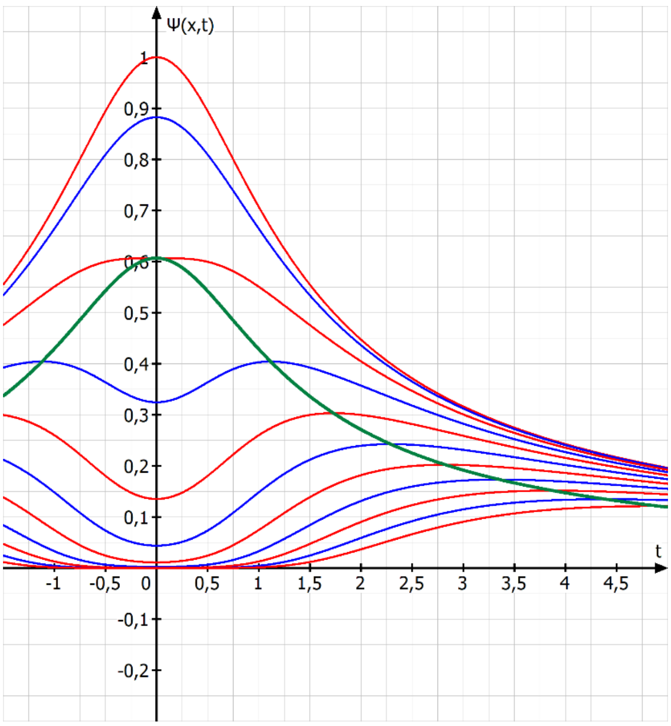
$$-\frac{t \cdot e^{-\frac{0.5*x^2}{t^2+1}}(1-x^2+t^2)}{(t^2+1)^{\frac{5}{2}}} = 0$$

We have a global maximum at $t = 0$.

… whatever this may mean …

We have local maxima at:

$x = 0$:	$1 + t^2 = 0 \rightarrow t = i$
$x = 1$:	$1 - 1 + t^2 = 0 \rightarrow t = 0$
$x = 2$:	$1 - 4 + t^2 = 0 \rightarrow t = \sqrt{3}$
$x = \cdots$	$t = \sqrt{x^2 - 1}$



The maxima are located at the green exponential, $f(t) = \frac{1}{\sqrt{1+t^2}}\exp(-\frac{1}{2})$.

What we see is propagating probability density.

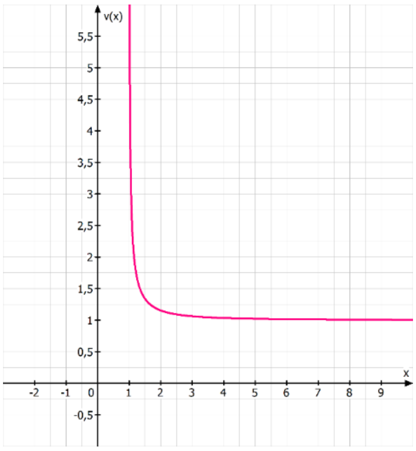
At position $x = 1$ we have the maximum at time $t = 0$.

At position $x = 1.5$ we have the maximum at time $t = \sqrt{x^2 - 1} = \sqrt{1.25}$.

We interpret this as a speed the maximum of the probability density is spreading out:

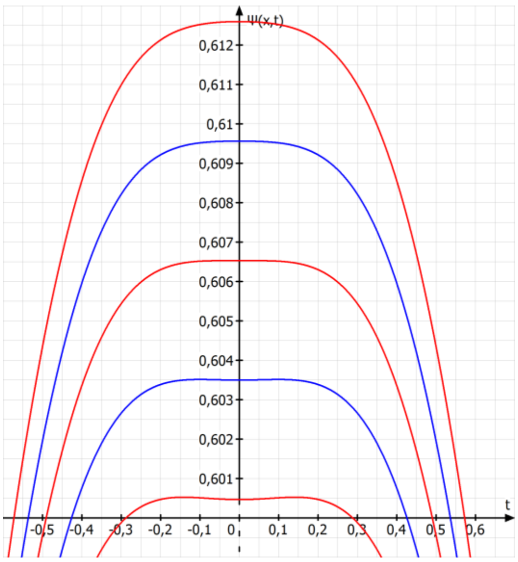
$$v = \frac{x}{t} = \frac{x}{\sqrt{x^2 - 1}} = \sqrt{\frac{1}{1 - \frac{1}{x^2}}}$$

If we are far enough from the origin, we get constant speed of the maximum-peak with decreasing amplitude.



If we examine the inner core, we can state the following:

- At position zero, we have constant change in the probability density.
- At position one we reach an area the probability density is nearly constant for a longer period and then constantly decreasing. We could describe this as a "hard shell" of the particle. To calculate the thickness correctly we would need to use all the constants given in the wave function.
- At a position greater than one we get the spreading maxima of the probability density.



Note: x (red/blue) in steps of 0.005 beginning with $x = 0.99$